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EDITED BY

FRANK MORLEY

WITH THE COOPERATION OF

SIMON NEWCOMB

A. COHEN, CHARLOTTE A. SCOTT

AND OTHER MATHEMATICIANS

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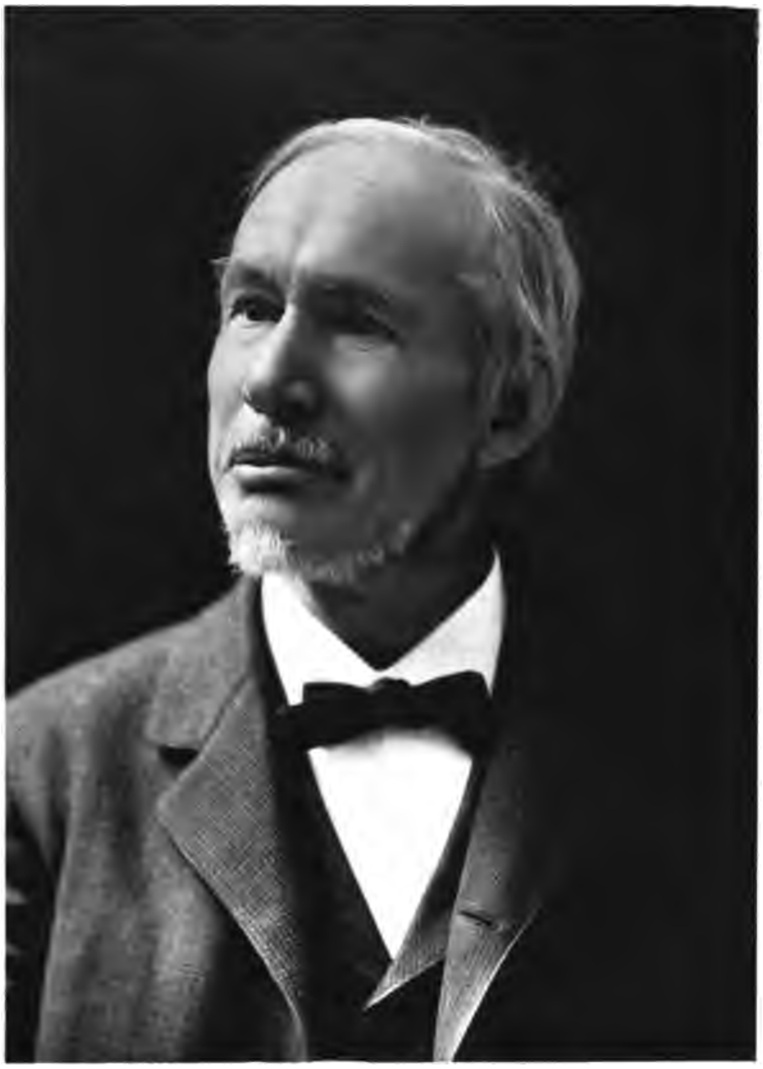

G. W. Hill.

Staven Fund

Some Properties of a Generalized Hypergeometric Function.

BY F. H. JACKSON.

In Heine's Kugelfunctionen, Vol. I, Appendix to Chapter 2, some properties of the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2 + \ldots \equiv \phi\,[a, b, c, q, x]$$

are given. The three-fold symmetry of this series is investigated by Professor L. J. Rogers, p. 171, Vol. XXIV, Proc. London Math. Soc. In this paper some properties of the more general series

$$1+\frac{[\alpha][\beta]}{[l]\,[\gamma]}\lambda x^{[l]}+\frac{[\alpha][\alpha+l][\beta][\beta+l]}{[l][2l][\gamma][\gamma+l]}\lambda^2 x^{[2l]}+\ldots \equiv F([\alpha][\beta][\gamma]\lambda x) \quad (1)$$

will be investigated.

Here $[\alpha]$ denotes $\dfrac{p^{\alpha}-1}{p-1}$.

Owing to the form of the indices of the element x, namely, $[l]\,.\,[2l]$, etc., a differential equation for F can be found analogous to the differential equation of the Hypergeometric Series:

F reduces to $\phi\,[a, b, c, q, x]$ if for p^{α} we substitute a;

p^{β}	"	"	b;
p^{γ}	"	"	c;
p^{l}	"	"	q;
λ	"	"	x;
x	"	"	1.

The differential equation satisfied by F is

$$
\begin{aligned}
\lambda[\alpha][\beta] \,.\, F + \{\lambda p^{\alpha}[\beta+l]\,x^{[l]} + \lambda p^{\beta}[\alpha]\,x^{[l]} - [\gamma]\}\, \frac{dF}{d(x)}& \\
+ \{\lambda p^{\alpha+\beta+l} x^{[2l]} - p^{\gamma} x^{p^l [l]}\}\, \frac{d^2F}{d^2(x)}& \\
= \lambda[\alpha][\beta]\,\{F([\alpha+l][\beta+l][\gamma] \,.\, \lambda \,.\, x)& \\
- F([\alpha+l][\beta+l][\gamma]\,\lambda \,.\, x^{p^l})\},& \qquad (2)
\end{aligned}
$$

in which

$$
\frac{d}{d(x)} \quad \text{is} \quad \frac{d}{d(x^{[l]})}, \qquad \frac{d^2}{d^2(x)} \quad \text{is} \quad \frac{d}{d(x^{p^l})} \cdot \frac{d}{d(x)},
$$

differentiation being with regard to $x^{[l]}$, $x^{p^l [l]}$, $x^{p^{2l} [l]}$, etc., successively as independent variables.

That F satisfies the above equation can be shown by a method analogous to the ordinary method for integrating equations in series. See Proc. London Math. Soc., series 2, Vol. I, page 83.

The principal results to be obtained in the following analysis are

$$
F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta}) = \frac{\Gamma([\gamma-\alpha-\beta])\,\Gamma([\gamma])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])}, \qquad (3)
$$

$$
F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l}) = \frac{\Gamma([\gamma-\alpha-\beta])\,\Gamma([\gamma])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])} \cdot \frac{\{[\gamma-\alpha-l] + p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-\beta-l]}; \qquad (4)
$$

Γ does not denote Euler's Gamma-Function but a certain convergent infinite product which reduces when $p = 1$ to Euler's Gamma-Function. Theorem (3) is well known and is due to Heine. The second theorem is, I think, new.

Differentiating the series $F([\alpha][\beta][\gamma] \,.\, \lambda x)$, term by term, we obtain

$$
\frac{dF}{d(x)} = \frac{[\alpha][\beta]}{[\gamma]}\,\lambda\, F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^{p^l}).
$$

Also $F([\alpha][\beta][\gamma]\,\lambda_1 x) - F([\alpha][\beta][\gamma-l]\,\lambda x)$ is a series in which the term involving $x^{[ml]}$ is

$$
\frac{[\alpha][\alpha+l]\ldots\ldots[\alpha+(m-1)l] \,.\, [\beta][\beta+l]\ldots\ldots[\beta+(m-1)l]}{[l][2l]\ldots\ldots[ml] \,.\, [\gamma][\gamma+l]\ldots\ldots[\gamma+(m-1)l]}
\times \lambda_1^m \left\{1 - \frac{\lambda^m}{\lambda_1^m} \cdot \frac{[\gamma+(m-1)l]}{[\gamma-l]}\right\} x^{[ml]}.
$$

If $\lambda_1 = \lambda$ this reduces to

$$-\frac{[\alpha][\alpha+l]\ldots\ldots[\alpha+(m-1)l]\,.\,[\beta][\beta+l]\ldots\ldots[\beta+(m-1)l]}{[l][2l]\ldots\ldots[(m-1)l]\,.\,[\gamma-l][\gamma]\ldots\ldots[\gamma+(m-1)l]}\,p^{\gamma-l}\lambda^m x^{[ml]},$$

which is the general term of

$$-\frac{p^{\gamma-l}[\alpha][\beta]}{[\gamma][\gamma-l]}\,\lambda x^{[l]}\,.\,F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^p),$$

that is of

$$-\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\,\frac{dF}{d(x)}.$$

So we have

$$F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha][\beta][\gamma-l]\,\lambda x) = -\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\cdot\frac{dF}{d(x)}. \qquad (5)$$

Similarly we may show that

$$F([\alpha][\beta][\gamma],\,\lambda p^l,\,x) - F([\alpha][\beta][\gamma-l]\,\lambda x) = -\frac{x^{[l]}}{[\gamma-l]}\cdot\frac{dF}{d(x)}. \qquad (6)$$

Now in the differential equation

$$\lambda[\alpha][\beta]F + \{\lambda p^{\alpha}[\beta+l]\,x^{[l]} + \lambda p^{\beta}[\alpha]\,x^{[l]} - [\gamma]\}\,\frac{dF}{d(x)}$$
$$+\{\lambda p^{\alpha+\beta+l}x^{[2l]} - p^{\lambda}x^{p^l[l]}\}\,\frac{d^2F}{d^2(x)}$$
$$= [\alpha][\beta]\,\lambda\,\{F([\alpha+l][\beta+l][\gamma]\,\lambda x) - F([\alpha+l][\beta+l][\gamma]\,\lambda x^{p^l})\}, \qquad (7)$$

putting $x = 1$ the right side vanishes identically and $\left[\dfrac{d^2F}{d^2(x)}\right]_{x=1}$ is finite, provided $p^l < 1$;

so choosing $\lambda = p^{\gamma-\alpha-\beta-l}$ to make the bracket coefficient of $\dfrac{d^2F}{d^2(x)}$ vanish we obtain from the differential equation

$$p^{\gamma-\alpha-\beta-l}[\alpha][\beta]\,F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})$$
$$+\{p^{\gamma-\beta-l}[\beta+l] + p^{\gamma-\alpha-l}[\alpha] - [\gamma]\}\left[\frac{dF}{d(x)}\right]_{x=1} = 0; \qquad (8)$$

putting $\lambda = p^{\gamma-\alpha-\beta-l}$ in (5) and eliminating $\left[\dfrac{dF}{d(x)}\right]_{x=1}$ between (5) and (8) we have a relation

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})$$
$$=\frac{[\gamma-l]\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-l][\gamma-\beta-l]}F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l}), \quad (9)$$

a relation between quasi-successive functions. This relation does not enable us to express $F([\alpha][\beta][\gamma]\ p^{\gamma-\alpha-\beta-l}$ in the form of an infinite product as would have been the case if the λ (namely $p^{\gamma-\alpha-\beta-l}$) in the F on the right side had been $p^{\gamma-\alpha-\beta-2l}$. When p is made unity $\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}$ reduces to $(\gamma-\alpha-\beta-l)$ and we have

$$F(\alpha\beta\gamma)=\frac{(\gamma-1)(\gamma-\alpha-\beta-1)}{(\gamma-\alpha-\beta)(\gamma-\beta-1)}F(\alpha\,.\,\beta\,.\,\gamma-1),$$

from which $F(\alpha\beta\gamma)$ may be expressed as an infinite product. To obtain a relation between two successive functions $F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})$ and $F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})$ we eliminate $\dfrac{dF}{d(x)}$ between

$$F([\alpha][\beta][\gamma]\,\lambda x) \qquad -F([\alpha][\beta][\gamma-l]\,\lambda x)=-\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\frac{dF}{d(x)}, \quad (10)$$

and

$$F([\alpha][\beta][\gamma],\,\lambda p^{l},\,x)-F([\alpha][\beta][\gamma-l]\,\lambda x)=-\frac{x^{[l]}}{[\gamma-l]}\frac{dF}{dx}, \quad (11)$$

and we obtain

$$(p^{\gamma-l}-1)\,F([\alpha][\beta][\gamma-l]\,\lambda,\,x)=p^{\gamma-l}F([\alpha][\beta][\gamma]\,p^{l}\lambda x)$$
$$-F([\alpha][\beta][\gamma]\,\lambda x)\,. \quad (12)$$

From the differential equation we have

$$\left[\frac{dF}{d(x)}\right]_{x=1}\times\{p^{\gamma-\beta-l}[\beta+l]+p^{\gamma-\alpha-l}[\alpha]-[\gamma]\}$$
$$+p^{\gamma-\alpha-\beta-l}[\alpha][\beta]\,F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})=0, \quad (13)$$

also we have from (6), putting $\lambda=p^{\gamma-\alpha-\beta-l}$,

$$\left[\frac{dF}{d(x)}\right]_{x=1}=[\gamma-l]\{F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})-F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})\}, \quad (14)$$

and from (12) we have, putting $\lambda=p^{\gamma-\alpha-\beta-l}$,

$$(p^{\gamma-l}-1)\,F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})$$
$$=p^{\gamma-l}F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})-F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l}). \quad (15).$$

Eliminating $\left[\dfrac{dF}{d(x)}\right]_{x=1}$ between equations (13) and (14), then eliminating

$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})$ by means of (15), we obtain

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta}) = \frac{[\gamma-l][\gamma-\alpha-\beta-l]}{[\gamma-\alpha-l][\gamma-\beta-l]}\,F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l}), \quad (15\text{A})$$

a relation between successive functions from which by repetition we have

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta}) = \underset{\kappa=\infty}{L}\frac{[\gamma-\alpha][\gamma-\alpha+l]\ldots[\gamma-\alpha+(k-1)l].[\gamma-\beta][\gamma-\beta+l]\ldots[\gamma-\beta+(k-1)l]}{[\gamma][\gamma+l]\ldots[\gamma+(k-1)l].[\gamma-\alpha-\beta]\ldots[\gamma-\alpha-\beta+(k-1)l]} \times F([\alpha][\beta][\gamma+kl]\,p^{\gamma-\alpha-\beta+\kappa l}). \quad (16)$$

Each of the four infinite products on the right side of the above is convergent if $p^l<1$ and $F([\alpha][\beta][\gamma+kl]\,p^{\gamma-\alpha-\beta+\kappa l})=1$ when k is infinite and $p^l<1$.

So we write

$$\frac{\Gamma([\gamma])\,\Gamma([\gamma-\alpha-\beta])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])} = F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})\,; \quad (17)$$

and by the help of (9) we deduce

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l} = \frac{\Gamma([\gamma])\,\Gamma([\gamma-\alpha-\beta])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])}\cdot\frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-\beta-l]}. \quad (18)$$

If we consider p^l not as a simple element but as compounded of the form

$$p_1^{l_1}p_2^{l_2}\ldots\ldots p_s^{l_s},$$

then p^{a+rl} is

$$p_1^{a_1+r l_1},\ p_2^{a_2+rl_2}\ldots\ldots p_s^{a_s+rl_s}.$$

The series F can be transformed into various series such as

$$1+p^{\gamma-\alpha-\beta}\frac{p^{\alpha}-1\,.\,p^{\beta}-z}{p^{l}-1\,.\,p^{\gamma}+z} + p^{2(\gamma-\alpha-\beta)}\frac{p^{\alpha}-1\,.\,p^{\alpha+l}-1\,.\,p^{\beta}+z\,.\,p^{\beta+l}+z}{p^{l}-1\,.\,p^{2l}-1\,.\,p^{\gamma}+z\,.\,p^{\gamma+l}+z}+\ldots\ldots, \quad (19)$$

$$1+p^{l}\frac{p^{\alpha}-1\,.\,p^{\beta}-1}{p^{l}-1\,.\,p^{\gamma}+z}z+p^{2l}\frac{p^{\alpha}-1\,.\,p^{\alpha+l}-1\,.\,p^{\beta}-1\,.\,p^{\beta+l}-1}{p^{l}-1\,.\,p^{2l}-1\,.\,p^{\gamma}+z\,.\,p^{\gamma+l}+z}z^{2}+\ldots\ldots \quad (20)$$

The following theorems obviously follow from the foregoing:

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})\,.\,F([-\alpha]\,[\beta]\,[\gamma-\alpha]\,p^{\gamma-\beta})=1, \quad (21)$$

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta}) \,.\, F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha}) = 1, \tag{22}$$

$$\begin{aligned} &F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})\,F([-\alpha][\beta][\gamma-\alpha]\,p^{\gamma-\beta-l}) \\ &\qquad = \frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}\{[\gamma-l]+p^{\gamma-\alpha-l}[-\beta]\}}{[\gamma-\alpha-\beta-l][\gamma-\beta-l]}, \end{aligned} \tag{23}$$

$$\begin{aligned} &F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})\,F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha-l}) \\ &\qquad = \frac{\{[\gamma-\beta-l]+p^{\gamma-l}[-\alpha]\}\{[\gamma-l]+p^{\gamma-\beta-l}[-\alpha]\}}{[\gamma-\alpha-\beta-l][\gamma-\alpha-l]}, \end{aligned} \tag{24}$$

$$\begin{aligned} &F([-\alpha][\beta][\gamma-\alpha]\,p^{\gamma-\beta-l}) \\ &\quad = \frac{[\gamma-\alpha-l]}{[\gamma-\beta-l]} \cdot \frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}\{[\gamma-l]+p^{\gamma-\alpha-l}[-\beta]\}}{\{[\gamma-\beta-l]+p^{\gamma-l}[-\alpha]\}\{[\gamma-l]+p^{\gamma-\beta-l}[-\alpha]\}} \\ &\qquad\qquad \times F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha-l}), \end{aligned} \tag{25}$$

$$p^{\alpha}[\beta]\{F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha][\beta+l][\gamma]\,\lambda x\}$$

$$= p^{\beta}[\alpha]\{F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha+l][\beta][\gamma]\,\lambda x)\} \tag{26}$$

$$= \lambda^{p^{\alpha+\beta}} \frac{[\alpha][\beta]}{[\gamma]}\, x^{[l]}\, F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^{p^l}) \tag{27}$$

$$= x^{[l]} p^{\alpha+\beta} \frac{dF}{d(x)}\,. \tag{28}$$

Other solutions of the differential equation give rise to the series

$$x^{[l-\gamma]}\, F([\alpha+l-\gamma][\beta+l-\gamma][2l-\gamma]\, x^{p^{l-\gamma}}\lambda),$$

and

$$\begin{aligned} &x^{[-\alpha]} + \frac{[\alpha][\gamma-\alpha-l]}{[l][\beta-\alpha-l]} \frac{p^{l-\alpha}}{\lambda} x^{[-l-\alpha]} \\ &\qquad + \frac{[\alpha][\alpha+l][\gamma-\alpha-l][\gamma-\alpha-2l]}{[l][2l][\beta-\alpha-l][\beta-\alpha-2l]} \cdot \frac{p^{2l-2\alpha}}{\lambda^2} x^{[-2l-\alpha]} + \ldots; \end{aligned}$$

the connection between the various solutions must be reserved for another paper.

Relation between Real and Complex Groups with Respect to their Structure and Continuity.

By Dr. S. E. Slocum.

Let G_r denote a given r-parameter group, generated by the r infinitesimal transformations $X_1, \ldots X_r$, where

$$X_j \equiv \sum_1^r {}_k \xi_{jk}(x_1 \ldots x_r)\frac{\partial}{\partial x_k}, \qquad (j = 1, 2, \ldots r).$$

If the finite equations defining a transformation T_a of this group are not the canonical equations of the group, the form of the infinitesimal transformation by which T_a is generated is not apparent.* It may, however, be obtained as follows: In order to transform the finite equations of T_a into their canonical form, it is necessary to introduce new parameters $\mu_1 \ldots \mu_r$ (the so-called canonical parameters) defined by equations of the form

$$\mu_k = N_k(a_1, a_2, \ldots a_r), \qquad (k = 1, 2, \ldots r),$$

which are obtained from the finite equations defining T_a by a process of differentiation, elimination and integration.† Then the transformation T_μ of the given group, in the canonical parameters $\mu_1 \ldots \mu_r$, is generated by the infinitesimal transformation

$$U_\mu \equiv \mu_1 X_1 + \ldots + \mu_r X_r.$$

Consequently, if we replace the μ's in this infinitesimal transformation by their functional values in terms of the a's, we obtain the infinitesimal transformation

---

* Man kann aber nicht so leicht einsehen welche infinitesimale Transformationen gerade eine der gefundenen endliche Transformationen erzeugt. Lie, Continuierliche Gruppen, p. 195.

† Lie, Transformations Gruppen, Vol. 3, pp. 609–11.

by which T_a is generated, namely,

$$U_a \equiv N_1(a) X_1 + \ldots\ldots + N_r(a) X_r.$$

For certain systems of values of the a's, say $\bar{a}_1 \ldots\ldots \bar{a}_r$, one or more of the functions $N_k(\bar{a})$ $(k = 1, 2, \ldots\ldots r)$ may be infinite in all branches. If such is the case, the transformation $U_{\bar{a}}$ is no longer infinitesimal, and, consequently, the transformation $T_{\bar{a}}$ is not generated by an infinitesimal transformation of the group. This is the briefest possible explanation of the well-known fact that discontinuity may occur in a group with continuous parameters.*

To illustrate what precedes, consider the finite equations

$$x_1' = x_1 e^{a_3} + a_1 e^{a_3}, \quad x_2' = x_2 + a_2, \quad x_3' = x_3 + a_3 \tag{1}$$

which define a transformation T_a of the group G_3 generated by the infinitesimal transformations whose symbols are $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$. Carrying out on T_a the process given by Lie for transforming these equations into their canonical form, we finally obtain the system of equations

$$\mu_1 = \frac{a_1 a_3 e^{a_3}}{e^{a_3} - 1} \equiv N_1(a), \quad \mu_2 = a_2 \equiv N_2(a), \quad \mu_3 = a_3 \equiv N_3(a),$$

which define the canonical parameters μ_1, μ_2, μ_3 in terms of a_1, a_2, a_3. A transformation T_μ of the group, in its canonical form, is generated by the infinitesimal transformation

$$U_\mu \equiv \mu_1 \frac{\partial}{\partial x_1} + \mu_2 \frac{\partial}{\partial x_2} + \mu_3 \left(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right).$$

Consequently, the transformation T_a, defined by equations (1); is generated by the infinitesimal transformation

$$U_a \equiv \frac{a_1 a_3 e^{a_3}}{e^{a_3} - 1} \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a \left(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right).$$

To verify this result, the finite equations generated by U_a can be obtained by summation of the infinite series

* Proc. Amer. Acad., Vol. 35, pp. 239–250, 483–485.

$$x_i' = x_i + \sum_j^r N^j(a) X_j x_i + \frac{1}{2!} \sum_{ij}^r \sum_k^r N_j(a)\, N_k(a)\, X_j X_k x_i + \ldots\ldots,$$
$$(i = 1,\ 2,\ \ldots\ldots\ r).$$

The equations resulting from this summation will then be found to be identical with equations (1) above. Let $\bar{a}_1 = a_1 \neq 0$, $\bar{a}_2 = a_2$, $\bar{a}_3 = 2k\pi\sqrt{-1}$, where k is any integer $\neq 0$. Then $N_1(\bar{a})$ is infinite in all branches. Consequently $U_{\bar{a}}$ is not infinitesimal, and, therefore, $T_{\bar{a}}$ is not generated by an infinitesimal transformation of the group.

Let ϕ denote the matrix of the bilinear form $-\sum_l^r \sum_k^r \left(\sum_j^r a_j c_{jkl}\right) y_l z_k$, where the e_{jkl} are the structural constants defining any given structure, and let Δ denote the determinant of the matrix $\frac{e^{\phi}-1}{\phi}$ (in which case $\Delta = \prod_k^r \frac{e^{\rho_k}-1}{\rho_k}$, where $\rho_1 \ldots\ldots \rho_r$ are the roots of the characteristic equation of ϕ). Then, if A_{kj} denotes the first minor of Δ relative to the constituent in the j^{th} row and k^{th} column, the infinitesimal transformation which generates the parameter group belonging to the structure with which Δ is associated, is defined by the equations

$$a_k' = a_k + \sum_{ij}^r \frac{A_{kj}}{\Delta} a_j \delta t, \qquad (k = 1,\ 2,\ \ldots\ldots\ r), \tag{2}$$

where a_k and a_k' $(k = 1,\ 2,\ \ldots\ldots\ r)$ are the variables which enter into the equations of the parameter group, a_k $(k = 1,\ 2,\ \ldots\ldots\ r)$ are its parameters and δt is an infinitesimal.*

In a previous article I have partially stated a theorem in regard to Δ which will now be completed. I have shown, namely, that if the determinant Δ associated with any given structure does not vanish for any system of values of the a's, all groups of the corresponding structure are continuous, whereas, if Δ vanishes for certain systems of values of a's, some groups of the corresponding structure may be continuous and others discontinuous.† In order to complete the theorem it is necessary to prove that if the a's can be so chosen that $\Delta = 0$, at least one group of the corresponding structure will be discontinuous. This is proved as follows: Since the parameters $a_1 \ldots\ldots a_r$ are independent, the first

* Proc. Amer. Acad., Vol. 36, p. 102.

† Ibid., Vol. 36, p. 101.

minors of Δ cannot all contain Δ as a factor, that is to say, the first minors of Δ cannot all contain all of the factors which enter into the product $\Delta = \prod_1^r {}_k \frac{e^{\rho_k}-1}{\rho_k}$. Consequently, if $\Delta = 0$, for certain values of the a's, one or more of the quotients $\frac{A_{kj}}{\Delta}$ must be infinite for these values of the a's. In this case the transformation (2) is no longer infinitesimal, and, therefore, the parameter group is discontinuous. If, then, $\bar{a}_1 \ldots . \bar{a}_r$ is a system of values of the a's for which $\Delta = 0$, and we apply a finite transformation of the parameter group to the point whose coordinates are $\bar{a}_1 \ldots . \bar{a}_r$, by properly choosing the parameters $\alpha_1 \ldots . \alpha_r$, this point can be transformed into any other finite point whatever of the manifold $(a_1 \ldots . a_r)$, but this finite transformation is not generated by an infinitesimal transformation of the group.

For example, consider the structure $(X_1, X_2) \equiv X_1$. The matrix ϕ in this case is

$$\phi \equiv \begin{vmatrix} a_2, & -a_1 \\ 0, & 0 \end{vmatrix}$$

and

$$\frac{e^\phi - 1}{\phi} \equiv \begin{vmatrix} \frac{e^{a_2}-1}{a_2}, & -\frac{a_1}{a_2^2}(e^{a_2}-a_2-1) \\ 0\ , & 1 \end{vmatrix}.$$

Therefore,

$$\Delta \equiv \frac{e^{a_2}-1}{a_2}.$$

[The roots of the characteristic equation of ϕ are $\rho_1 = a_2$, $\rho_2 = 0$, and, therefore, we also have $\Delta = \prod_1^r {}_k \frac{e^{\rho_k}-1}{\rho_k} \equiv \frac{e^{a_2}-1}{a_2}$]. Consequently, the equations defining the infinitesimal transformation of the parameter group belonging to the above structure are

$$\left.\begin{aligned} a_1' &= a_1 + \frac{a_2}{e^{a_2}-1}\,\alpha_1\delta t + \frac{a_1(e^{a_2}-a_2-1)}{a_2(\ e^{a_2}-1)}\,\alpha_2\delta t, \\ a_2' &= a_2 + \alpha_2\delta t, \end{aligned}\right\} \tag{3}$$

and the finite equations of this parameter group are found to be

$$\left.\begin{aligned} a_1' &= \frac{a_2+\alpha_2}{e^{a_2+\alpha_2}-1}\left[e^{\alpha_2}\,\frac{a_1}{a_2}\,(e^{a_2}-1) + \frac{\alpha_1}{\alpha_2}\,(e^{\alpha_2}-1)\right], \\ a_2' &= a_2 + \alpha_2. \end{aligned}\right\} \tag{4}$$

If we apply this transformation to the point whose coordinates are $a_1 = a_1$, $a_2 = 2k\pi\sqrt{-1}$, where k is any integer $\neq 0$, by a proper choice of the parameters a_1, a_2, this point can be transformed into any finite point whatever of the manifold (a_1, a_2). But for these values of the a's, $\Delta = 0$; consequently, the transformation (3) is not infinitesimal, and, therefore, the finite transformation (4) is not generated by an infinitesimal transformation of the group.

To each r-parameter complex group, G_r, there corresponds a definite r-parameter real group g_r, the properties of which are closely related to those of G_r.* It is possible, however, for g_r to be continuous and G_r discontinuous.† But if a group is continuous, two points of general position on the same smallest invariant manifold relative to that group can always be *continuously* interchanged by the transformations of the group, whereas Rettger has shown that if a group is discontinuous, two such points cannot always be so interchanged; that is to say, cannot always be interchanged by a transformamation of the group which can be generated by an infinitesimal transformation of the group.‡ For all the discontinuous complex groups, G_r, cited by Rettger to illustrate this statement, the corresponding real groups, g_r, are continuous. Thus it appears that the exception imposed upon Lie's chief theorem by the possibility of discontinuity in a group with continuous parameters necessitates a restriction upon the relation between the transitivity of a complex group and that of its corresponding real group.

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* Lie, Transformationsgruppen, Vol. 3, p. 362.

† E. g. consider the three-parameter group defined by the equations

$$\begin{aligned} x_1' &= x_1 e^{a_1} + a_1^2 e^{a_1} x_2 + 2a_1 e^{a_1} x_3 + a_2, \\ x_2' &= x_2 e^{a_1}. \\ x_3' &= a_1 e^{a_1} x_2 + e^{a_1} x_3 + a_3. \end{aligned}$$

The infinitesimal transformation which generates the finite transformation T_a defined by these equations, is found to be

$$U_a \equiv a_1 \left(2x_3 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) + \frac{a_1}{e^{a_1} - 1} \left(a_2 + 2a_3 - \frac{2a_1 a_3 e^{a_1}}{e^{a_1} - 1} \right) \frac{\partial}{\partial x_1} + \frac{a_1 a_3}{e^{a_1} - 1} \frac{\partial}{\partial x_3}.$$

For all real values of the a's, U_a is infinitesimal, and, consequently, the real group g_r, of transformations T_a is continuous. But for the complex values $a_1 = 2k\pi\sqrt{-1}$, $a_2 = a_2$, $a_3 = a_3 \neq 0$, where k is any integer $\neq 0$, U_a is no longer infinitesimal, and, consequently, the complex group, G_r, of transformations T_a is discontinuous.

‡ Amer. Jour., Vol. XXII, pp. 90–94.

Several structures which are of the same type for complex groups may constitute entirely distinct types of structure for real groups.* This follows from the fact that for real groups the structural constants defining any given structure must all be real, and two real structures can only be said to belong to the same type, when one can be transformed into the other by a real transformation.†

Suppose, then, that we have given two structures which are of the same type, A, for complex groups, but which constitute distinct types, B and C, for real groups. If the determinant Δ, associated with type A, does not vanish for any system of values of the parameters, *all* groups of this type are continuous, and, consequently, all real groups of types B and C are continuous as well. If, however, the determinant Δ, associated with type A, vanishes for certain systems of values of the parameters, one of the following cases may occur with respect to the continuity of the real groups of types B and C:

1. All real groups of both types B and C are continuous.

2. All real groups of both types B and C are discontinuous.

3. All real groups of one type, B, are continuous, and one or more, but not all, real groups of the other type, C, are discontinuous.

4. All real groups of one type, B, are continuous, and all real groups of the other type, C, are discontinuous.

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* E. g. consider the four structures

$$1.\ \begin{cases} (X_1, X_2) \equiv X_3, & (X_1, X_3) \equiv -X_2, \quad (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv 0, & (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0. \end{cases}$$

$$2.\ \begin{cases} (X_1, X_2) \equiv X_1, & (X_1, X_3) \equiv 2X_2, \quad (X_2, X_3) \equiv X_3, \\ (X_1, X_4) \equiv 0, & (X, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0. \end{cases}$$

$$3.\ \begin{cases} (X_1, X_2) \equiv 0, & (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_4, \\ (X_1, X_4) \equiv 0, & (X_2, X_4) \equiv X_1 + X_3, \quad (X_3, X_4) \equiv X_2. \end{cases}$$

$$4.\ \begin{cases} (X_1, X_2) \equiv -X_4 & (X_1, X_3) \equiv -X_4, \quad (X_2, X_3) \equiv 0, \\ (X_1, X_4) \equiv X_3, & (X_2, X_4) \equiv X_1, \quad (X_3, X_4) \equiv X_1. \end{cases}$$

It will be found that each of the above structures can be transformed into any one of the others, but only by means of a complex transformation. Thus the transformation

$$X_1 \equiv -X_1' - X_3'\sqrt{-1}, \quad X_2 \equiv X_2'\sqrt{-1}, \quad X_3 \equiv X_3'\sqrt{-1} - X_1', \quad X_4 \equiv X_4'$$

transforms (1) into (2), the transformation

$$X_1 \equiv X_4', \quad X_2 \equiv X_2', \quad X_3 \equiv \frac{\sqrt{-1}}{2}(X_1' - X_3') - X_4', \quad X_4 \equiv -\frac{\sqrt{-1}}{2}(X_1' + X_3')$$

transforms (2) into (3), etc. Consequently, the above four structures are of the same type for complex groups but constitute distinct types of structure for real groups.

† Transformations gruppen, Vol. 3, p. 361; Proc. Amer. Acad.. Vol. 36, p. 105.

The following four examples show the possibility of the occurrence of each of the above four cases:

Case 1. Consider the structures

$$\begin{cases}(X_1, X_2) \equiv 0 \quad , & (X_1, X_3) \equiv 0 \quad , \quad (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv 2X_1, & (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 2X_2 + X_3,\end{cases}$$

and

$$\begin{cases}(X_1, X_2) \equiv 0 \quad , & (X_1, X_3) \equiv X_1 + 2X_4, \quad (X_2, X_3) \equiv 2X_2, \\ (X_1, X_4) \equiv X_2, & (X_2, X_4) \equiv 0 \qquad\qquad , \quad (X_3, X_4) \equiv - X_4.\end{cases}$$

These are of the same type, A, for complex groups but constitute distinct types, B and C, for real groups, since they can only be interchanged by means of a complex transformation. The adjoined complex group of type A is discontinuous, and, consequently, all complex groups of that type are discontinuous.* The determinants Δ_1 and Δ_2, associated with these two structures, are respectively

$$\Delta_1 \equiv \frac{(e^{a_4} - 1)^2(e^{2a_4} - 1)}{2a_4^3}, \quad \Delta_2 \equiv \frac{(e^{a_3} - 1)^2(e^{2a_3} - 1)}{2a_3^3}.$$

Since neither of these determinants vanishes for any system of real values of the a's, all real groups of both types B and C are continuous.

Case 2. Consider the structures

$$(X_1, X_2) \equiv X_3 \quad , \quad (X_1, X_3) \equiv - X_2, \quad (X_2, X_3) \equiv X_1 \quad ,$$

and

$$(X_1, X_2) \equiv - 2X_1, \quad (X_1, X_3) \equiv X_2 \quad , \quad (X_2, X_3) \equiv - 2X_3,$$

which are of the same type for complex groups but constitute distinct types for real groups. The adjoined complex group of type A is discontinuous, and, consequently, all complex groups of this type are discontinuous. The determinants Δ_1 and Δ_2, associated with these two structures, are respectively

$$\Delta_1 \equiv \frac{(e^{\sqrt{-(a_1^2 + a_2^2 + a_3^2)}} - 1)(e^{-\sqrt{-(a_1^2 + a_2^2 + a_3^2)}} - 1)}{a_1^2 + a_2^2 + a_3^2},$$

$$\Delta_2 \equiv \frac{(e^{2\sqrt{a_2^2 + a_1 a_3}} - 1)(e^{-2\sqrt{a_2^2 + a_1 a_3}} - 1)}{- 4\,(a_2^2 + a_1 a_3)}.$$

Δ_1 vanishes if $a_1^2 + a_2^2 + a_3^2 = 4k^2\pi^2$; Δ_2 vanishes if $a_2^2 + a_1 a_3 = - k^2\pi^2$, where k is any integer $\neq 0$, and each of these relations can evidently be satisfied by real values of a_1, a_2, a_3. Moreover, the adjoined real groups of both types B and C are discontinuous, and, consequently, all real groups of both of these types are discontinuous.

* Taber, Bull. Amer. Math. Soc., Feb., 1900, p. 208.

Case 3. The structures

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_2, \quad (X_2, X_3) \equiv -X_1,$$

and

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_2$$

can also be shown to be of the same type, A, for complex groups, but to constitute distinct types, B and C, for real groups.

The determinant Δ, associated with type A, vanishes for certain systems of complex values of the a's, but the adjoined complex group of this type is continuous. Consequently, one or more, but not all, complex groups of type A are discontinuous. The determinants Δ_1 and Δ_2, associated with types B and C, are respectively

$$\Delta_1 \equiv \frac{(e^{a_3\sqrt{-1}} - 1)(e^{-a_3\sqrt{-1}} - 1)}{a_3^2}, \quad \Delta_2 \equiv \left(\frac{e^{a_3} - 1}{a_3}\right)^2.$$

Δ_1 vanishes for the real values $a_3 = 2k\pi$, where k is any integer $\neq 0$, but the adjoined real group of this type is continuous. Consequently, one or more, but not all, real groups of this type are discontinuous. Δ_2 does not vanish for any system of real values of the parameters, and, consequently, all real groups of the second type of structure are continuous.

Case 4. Consider structures 3 and 4 in the foot-note to page 12, which were there shown to be of the same type, A, for complex groups, but to constitute distinct types, B and C, for real groups. The adjoined complex group of type A is discontinuous; consequently, all complex groups of this type are discontinuous. The determinants Δ_3 and Δ_4, associated with the structures under consideration, are respectively

$$\Delta_3 \equiv \frac{(e^{a_3\sqrt{-1}} - 1)(e^{-a_3\sqrt{-1}} - 1)}{a_3^2}, \quad \Delta_4 \equiv \frac{(e^{a_4} - 1)(e^{-a_4} - 1)}{-a_4^2}.$$

Δ_4 does not vanish for any system of real values of the a's, and, therefore, all real groups of type 4 are continuous. Δ_3, however, vanishes for the real values $a_3 = 2k\pi$, where k is any integer $\neq 0$. Moreover, the real adjoined group of type 3 is discontinuous, and, therefore, all real groups of this type are discontinuous.*

* The symbols of infinitesimal transformation of this real adjoined group are

$$-x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_3}, \quad x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2}, \quad x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_2}.$$

UNIVERSITY OF CINCINNATI.

Determination of all the Characteristic Subgroups of any Abelian Group.

By G. A. Miller.

§1.—*Statement of the Problem and of the Principal Results.*

A subgroup which corresponds to itself in every possible simple isomorphism of the group (G) with itself, is called a characteristic subgroup.* The term was first used by Frobenius in Berliner Sitzungsberichte, 1895, p. 183. Burnside observed that a group which has no characteristic subgroup is either simple or the direct product of simply isomorphic simple groups.† The present paper is devoted to a determination of all the possible characteristic subgroups of G whenever G is abelian. In what follows it will be assumed that all the groups under consideration are abelian.

In every possible simple isomorphism of G with itself, every Sylow subgroup‡ must correspond to itself. Hence, it follows that the necessary and sufficient condition that a subgroup of G is characteristic is that each of the Sylow subgroups of this subgroup is characteristic in a Sylow subgroup of G. If we include the identity and G itself in the term characteristic subgroup, it follows that the number of the characteristic subgroups of G is the product of the numbers of the characteristic subgroups in all the Sylow subgroups of G. That is, the characteristic subgroups of G are the direct products of characteristic sub-

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* A characteristic subgroup may also be defined as a subgroup which is transformed into itself by all the operators in the group of isomorphisms of G.

† Burnside, Theory of Groups of Finite Order. 1897, p. 232.

‡ If the order of a group is divisible by p^m but not by p^{m+1}, a subgroup of order p^m, is called a Sylow subgroup. The term was first used in Bulletin of the American Mathematical Society, Vol. 9 (1903), p. 543.

groups of the Sylow subgroups of G. Hence we may confine our attention to the case where the order of G is p^m, p being any prime number. This will be done in what follows unless the contrary is explicitly stated.

The principal results are as follows:

When p is odd, two characteristic subgroups cannot be of the same type, and the number of characteristic subgroups in G is equal to the number of its sets of operators such that each set is composed of all the operators which are conjugate under the group of isomorphisms (I) of G. All the possible characteristic subgroups (besides the identity *) have a certain characteristic subgroup (C_1) in common. This is called the *fundamental characteristic subgroup* of G and is the only one in which all the operators besides the identity are conjugate under I. If a characteristic subgroup includes an operator (s) which could be used as an independent generator of G, then it includes all the operators of G whose orders divide the order of s. In particular, if a characteristic subgroup includes an operator of highest order, it must be G itself. If a characteristic subgroup includes k invariants, which are equal to or greater than p^δ, it must include all the characteristic subgroups which do not have more than k invariants, provided none of these invariants exceeds p^δ. The total number of the characteristic subgroups of G can be directly obtained by means of a simple formula.

§2.—*The Characteristic Subgroups which are Generated by Operators of Order p.*

Although the results of this section are special cases of those which will be developed in the following sections, yet it seems desirable to give them separately on account of their simplicity and their fundamental importance in the theory of characteristic subgroups. The treatment of the following sections can be presented in a somewhat briefer form in view of these special developments.

Suppose that β_1 invariants of G are equal to p^{α_1}, β_2 are equal to p^{α_2}, ..., β_λ are equal to p^{α_λ}; where $\alpha_1 > \alpha_2 > \dots > \alpha_\lambda > 0$. Hence $\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_\lambda\beta_\lambda = m$. Representing a set of independent generators corresponding to

* In what follows, the identity will not be included in the term characteristic subgroup, but G will be regarded as a characteristic subgroup of itself.

these invariants, in order, by $s_1, s_2, \ldots, s_{\beta_1+\beta_2+\ldots+\beta_\lambda}$, the group G may be regarded as the direct product of the subgroups

$$H_1, H_2, \ldots, H_\lambda,$$

which are respectively generated* by the independent generators of the same order, as follows:

$$H_1 \equiv \{s_1, s_2, \ldots, s_{\beta_1}\}, \quad H_2 \equiv \{s_{\beta_1+1}, s_{\beta_1+2}, \ldots, s_{\beta_1+\beta_2}\} \ldots$$
$$H_\lambda \equiv \{s_{\beta_1+\beta_2+\ldots+\beta_{\lambda-1}+1}, s_{\beta_1+\beta_2+\ldots+\beta_{\lambda-1}+2}, \ldots, s_{\beta_1+\beta_2+\ldots+\beta_\lambda}\}.$$

All the operators of the same order in H_{λ_1}, $\lambda_1 \overline{\gtrless} \lambda$, are conjugate under its group of isomorphisms (I_{λ_1}), since each of these operators is the same power of operators which could be selected separately as independent generators of H_{λ_1}. These operators are also conjugate under I since I includes the direct product of $I_1, I_2, \ldots, I_\lambda$.† If we multiply any operator of order p^{e_1} in H_1 by any operator in $\{H_2, H_3, \ldots, H_\lambda\}$ and raise the product to the p^{e_1-1} power, the constituent from $\{H_2, H_3, \ldots, H_\lambda\}$ will reduce to the identity. That is, H_1 includes all the operators of order p in G which are found in cyclic subgroups of order p^{e_1}, and it contains no other operators of order p. In other words, the identity and the operators of order p in H_1 constitute a characteristic subgroup (C_1') of G. It will soon be proved that $C_1' \equiv C_1$.

The group $\{H_1, H_2\}$ contains all the operators of order p in G which are contained in cyclic subgroups of order p^{e_2}, and all of its operators of order p have this property. Hence these operators, together with the identity, constitute a second characteristic subgroup (C_2) of G. All the operators of C_2, which are not also in C_1, are conjugate under I, since the independent generators of H_2 might have been so selected that the p^{e_2-1} power of one of them would give any given one of these operators. Since a characteristic subgroup may be defined as a subgroup which includes all the conjugates under I of its own operators, and since a group may always be generated by the operators which do not belong to any given subgroup, it follows that every characteristic subgroup which involves an operator of order p from C_2 must include C_1'.

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* This symbol denotes the group generated by all the operators inclosed.

† Transactions of the American Mathematical Society, Vol. 1 (1900), p. 396.

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In an exactly similar manner we observe that all the operators of order p in $\{H_1, H_2, \ldots, H_\epsilon\}$, $1 < \epsilon \overline{\gtrless} \lambda$, are composed of the operators of order p in the cyclic subgroups of G whose order is p^{α_ϵ}. Hence, these operators, together with the identity, constitute a characteristic subgroup (C_ϵ), which includes $C_1', C_2, \ldots, C_{\epsilon-1}$. All the operators of C_ϵ, which are not also in $C_{\epsilon-1}$, are conjugate under I because each of them is the $p^{\alpha_\epsilon - 1}$ power of an operator which could have been selected as an independent generator of H_ϵ. Hence, there are just λ characteristic subgroups of G which are generated by operators of order p. The number of invariants of these subgroups are respectively $\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \ldots + \beta_\lambda$. Since every characteristic subgroup must involve operators of order p, and since C_ϵ includes $C_{\epsilon-1}$, it follows that C_1' *must be contained in every possible characteristic subgroup of* G. That is, $C_1' \equiv C_1$.

§3.—*The Conjugates under I of any Operator of G.*

Let s represent any operator of order p^δ in H_γ $(\delta \overline{\gtrless} \alpha_\gamma, \gamma \overline{\gtrless} \lambda)$. It has been observed that all the operators of order p^δ in H_γ are conjugate under I. We proceed to find all the other conjugates of s under I. Consider all the products obtained by multiplying all the operators of order p^δ in H_γ into all the operators whose orders do not exceed p^δ in $\{H_1, H_2, \ldots, H_{\gamma-1}\}$. As all these products are the $p^{\alpha_\gamma - 1}$ power of operators which could be chosen separately as independent generators of G, they must be conjugate under I. Since G may be regarded as the direct product of the three groups $\{H_1, H_2, \ldots, H_{\gamma-1}\}$, H_γ, $\{H_{\gamma+1}, H_{\gamma+2}, \ldots, H_\lambda\}$, it remains only to find the operators in the last one of these groups which are such that the products obtained by multiplying them into an operator of order p^δ in H_γ are conjugate with s. This follows directly from the fact that all the conjugates of s under I must be such that their $p^{\delta-1}$ power is in C_γ but not in $C_{\gamma-1}$.

Since s is the $p^{\alpha_\gamma - \delta}$ power of an operator which may be used as an independent generator of G, each of its conjugates must have the same property. Hence, the factor from $H_{\gamma+\gamma_1}$ $(\gamma_1 = 1, 2, \ldots, \lambda - \gamma)$ may be any operator whose order does not exceed $p^{\alpha_{\gamma+\gamma_1} - \alpha_\gamma + \delta}$ $(\alpha_{\gamma+\gamma_1} - \alpha_\gamma + \delta > 0)$, but no other operator of $H_{\gamma+\gamma_1}$ can be multiplied into s without affecting the system of conjugates to which s belongs, since such a product would not be the $p^{\alpha_\gamma - \delta}$ power of

an operator which could be used as an independent generator* of G. Hence, *all the conjugates of s under I can be obtained by multiplying all the operators of order p^{δ} in H_{γ} by the group generated by all the operators whose orders divide p^{δ} in $H_1, H_2, \ldots, H_{\gamma-1}$, together with the operators whose orders divide $p^{\alpha_{\gamma+\gamma_1}-\alpha_{\gamma}+\delta}$ in $H_{\gamma+\gamma_1}$.* The number of these conjugates is, therefore, $p^{l}(p^{\beta_{\gamma}}-1)$, where

$$l=\delta(\beta_1+\beta_2+\ldots+\beta_{\gamma})-\beta_{\gamma}+\sum_{\gamma_1=1}^{\gamma_1=\lambda-\gamma}\beta_{\gamma+\gamma_1}(\alpha_{\gamma+\gamma_1}-\alpha_{\gamma}+\delta),$$
$$\alpha_{\gamma+\gamma_1}-\alpha_{\gamma}+\delta>0.$$

That is, the last summation is to extend only to such values of γ_1 as will make $\alpha_{\gamma+\gamma_1}-\alpha_{\gamma}+\delta>0$. In other words, only the positive terms of the series are to be combined. In particular, when $\delta=\alpha_{\gamma}$, we arrive at the formula of the last foot-note.

The preceding method can be employed to determine all the conjugates under I of operators which are powers of possible independent generators. All the operators of orders p and p^{α_1} in G possess this property. The necessary and sufficient condition that all the operators of G are powers of possible independent generators is that either $\lambda=1$ or that both of the following conditions are satisfied: $\lambda=2$, $\alpha_1-\alpha_2=1$. We proceed to consider the number of conjugates of an operator (t) of G when t is not some power of a possible independent generator of G.

The constituents of t which belong to $H_1, H_2, \ldots, H_{\lambda}$ may be denoted respectively by $t_1, t_2, \ldots, t_{\lambda}$ and their orders by $p^{\alpha_1'}, p^{\alpha_2'}, \ldots, p^{\alpha_{\lambda}'}$. If the order of t is p^{δ}, at least one of the numbers $\alpha_1', \alpha_2', \ldots, \alpha_{\lambda}'$ must be equal to δ, while each of the others is equal to or less than δ. Let γ be the largest subscript of a constituent of order p^{δ}. From the fact that the constituents of t are separately powers of possible independent generators of G, it follows that there

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* Any operator of order p^{δ} in any abelian group whatever can be used as an independent generator, provided its $p^{\delta-1}$ power is not included in some cyclic subgroup of order $p^{\delta+1}$. Hence, all the operators of order $p^{\alpha_{\gamma}}$ which can be used as independent generators of G are the products of operators of highest order in H_{γ} into the group generated by all the operators of $\{H_{\gamma+1}, H_{\gamma+2}, \ldots, H_{\lambda}\}$ and the operators of $\{H_1, H_2, \ldots, H_{\gamma-1}\}$ whose orders divide $p^{\alpha_{\gamma}}$. The number of these operators is

$$\{p^{\beta_{\gamma}\alpha_{\gamma}}-p^{\beta_{\gamma}(\alpha_{\gamma}-1)}\}p^{\alpha_{\gamma}(\beta_1+\beta_2+\ldots+\beta_{\gamma-1})+\beta_{\gamma+1}\alpha_{\gamma+1}+\beta_{\gamma+2}\alpha_{\gamma+2}+\ldots+\beta_{\lambda}\alpha_{\lambda}}.$$

is at least one conjugate of t such that both of the following conditions are satisfied:

$$\alpha'_\rho \overline{\gtrsim} \alpha'_{\rho+1}, \quad \alpha_\rho - \alpha'_\rho \overline{\gtrsim} \alpha_{\rho+1} - \alpha'_{\rho+1}, \qquad \rho = 1, 2, \ldots, \lambda - 1.$$

The necessary and sufficient condition that t is not a power of some operator which could be used as an independent generator of G is that $\alpha'_{\gamma+\gamma_1} > \alpha_{\gamma+\gamma_1} - \alpha_\gamma + \delta$ for at least one value of $\gamma_1 = 1, 2, \ldots, \lambda - \gamma$. We shall suppose that t satisfies all of these conditions. In other words, we shall suppose that t was so selected that it is not a power of some operator which could be used as an independent generator of G, and also that the order of each of its constituents is at least as large as the order of the corresponding constituent in any one of its conjugates.

All the conjugates of t can be obtained from the fact that all the conjugates of its constituents are known. From the preceding paragraphs, it can be seen that the following method gives all these conjugates: Construct the group formed by all the operators whose orders divide $p^{\alpha'_\rho}$ in the constituents in which at least one of the following two conditions is satisfied:

$$\alpha'_\rho = \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} = \alpha_\rho - \alpha'_\rho,$$

and multiply each of the operators of this group into all the operators obtained by multiplying together all the operators of order $p^{\alpha'_\rho}$ in the constituents for which both of the following conditions are satisfied:

$$\alpha'_\rho > \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} > \alpha_\rho - \alpha'_\rho.$$

If t is regarded as any operator of G, all the conjugates of t must involve at least one constituent whose order is the same in all of the conjugates. If it involves only one such constituent, t is a power of some operator which could be used as an independent generator of G; if it involves more than one such constituent, then t cannot be a power of a possible independent generator. It is also clear that the number of conjugates of an operator which is not a power of a possible independent generator is always larger than the number of conjugates of some operator of the same order which is a power of a possible independent generator. For the present purpose it is only necessary to bear in mind that the given method leads to all the conjugates of t under I.

The group generated by all the conjugates of t is evidently the direct product of all the groups formed by all the operators whose orders divide $p^{\alpha'_\rho}$ in H_ρ $(\rho = 1, 2, \ldots, \lambda)$ whenever $p > 2$.* When $p = 2$, this result remains true only if there is no more than one cyclic H_ρ for which both of the conditions

$$\alpha'_\rho > \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} > \alpha_\rho - \alpha'_\rho$$

are satisfied. If there are $k > 1$ such constituents when $p = 2$, the group generated by the conjugates of t is the direct product of the group obtained by dimidiating† the cyclic subgroups of order $p^{\alpha'_\rho}$ of all these k constituents, and the subgroups which are composed of all the operators whose orders divide $p^{\alpha'_\rho}$ in the other H_ρ's. The order of the group generated by the conjugates of t is therefore

$$p^{\alpha'_1\beta_1 + \alpha'_2\beta_2 + \ldots + \alpha'_\lambda\beta_\lambda}$$

except when both of the conditions $p = 2$, $k > 1$ are satisfied. In this case the order is this number divided by 2^{k-1}.

§4.—*Determination of all the Characteristic Subgroups of G.*

The simplest case presents itself when all the invariants of G are equal; that is, when $G \equiv H_1$. It was observed above that all the operators of the same order are conjugate under the I of such a G, and hence there are just α_1 characteristic subgroups. These are composed respectively of all the operators of G whose orders divide p^β, $\beta = 1, 2, \ldots, \alpha_1$. Each of them contains just β_1 equal invariants and is of order $p^{\beta\beta_1}$. It is evident that all the operators whose orders divide p^β must always constitute a characteristic subgroup of G and that G con-

* This result follows directly from the facts that the conjugates of t must generate a group which can be obtained by establishing some isomorphism between the subgroups of the H_ρ's, which are composed of the operators whose orders divide $p^{\alpha'_\rho}$, and that all the operators of such a subgroup must correspond to a given operator whenever all the operators of the highest order have this property except when both H_ρ is cyclic and $p = 2$.

† Cayley, Quarterly Journal of Mathematics, Vol. 25 (1890), p. 71.

tains additional characteristic subgroups whenever it has at least two unequal invariants.*

We proceed to determine all the characteristic subgroups of G which involve C_ρ, but do not include $C_{\rho+1}$, $0 < \rho \overline{\gtrless} \lambda$. Let C be such a characteristic subgroup, and suppose that t is an operator of C which has been so selected that each of its constituents $t_1, t_2, \ldots, t_\lambda$ from $H_1, H_2, \ldots, H_\lambda$ respectively is of the largest possible order. As in the preceding section, we may suppose that the orders of these constituents (which exceed unity) are $p^{\alpha'_1}, p^{\alpha'_2}, \ldots, p^{\alpha'_\rho}$ respectively. Since C may be defined as any subgroup of G which includes all the conjugates under I of each one of its operators, it is clearly generated by the conjugates of t and its order is

$$p^{\alpha'_1\beta_1 + \alpha'_2\beta_2 + \ldots + \alpha'_\rho\beta_\rho}$$

whenever $p > 2$. That is, *when* $p > 2$, *each of the characteristic subgroups of* G *is generated by one and only one complete set of conjugate operators under* I, *and the independent generators of such a subgroup may be so selected that they are powers of the independent generators of* G.

All the C's in question can be obtained as follows:

The ρ invariants may all be equal to p. Then the first invariant may be successively multiplied by p without affecting the others until it is equal to $p^{\alpha_1-\alpha_2+1}$. After this, the second invariant may be made equal to p^2 and the first invariant may be successively increased from p^2 to $p^{\alpha_1-\alpha_2+2}$, etc. To each such set of invariants there corresponds a C. Hence, the number of the characteristic subgroups of G which contain C_ρ without also containing $C_{\rho+1}$ is

$$(\alpha_1 - \alpha_2 + 1)(\alpha_2 - \alpha_3 + 1) \ldots (\alpha_\rho - \alpha_{\rho+1})$$

whenever $p > 2$. In this case the total number of characteristic subgroups in

* It may be observed that $G \equiv H_1$ whenever all the operators of order p^δ in G are conjugate under I for at least two values of $\delta > 0$. That is, while all the operators of highest order in G are always conjugate under I, all those of any other order are never conjugate unless all the invariants of G are equal.

G is given by the simple formula

$$\sum_{\rho=1}^{\rho=\lambda}(\alpha_1-\alpha_2+1)(\alpha_2-\alpha_3+1)\ldots\ldots(\alpha_{\rho-1}-\alpha_\rho+1)(\alpha_\rho-\alpha_{\rho+1}),\quad \alpha_{\lambda+1}=0.$$

This formula remains true when $p=2$ if there is no more than one cyclic $H_l\,(l \gtrless \rho)$ for which both of the conditions

$$\alpha'_\rho > \alpha'_{\rho+1},\quad \alpha_{\rho-1}-\alpha'_{\rho-1} > \alpha_\rho-\alpha'_\rho$$

are satisfied. If there are $k>1$ such cyclic H_l's, there are as many additional C's which correspond to the conjugates of such a t as there are combinations of k things taken 2, 3, ..., k at a time. None of these is generated by the conjugates of t. It should be observed that *the formula*

$$\sum_{\rho=1}^{\rho=\lambda}(\alpha_1-\alpha_2+1)(\alpha_2-\alpha_3+1)\ldots\ldots(\alpha_{\rho-1}-\alpha_\rho+1)(\alpha_\rho-\alpha_{\rho+1})$$

gives the number of sets of operators in G which are conjugate under I regardless of the value of p. The identity is not included in this enumeration of sets of conjugate operators.

Among the special results, it may be well to observe that every characteristic subgroup which contains an operator of order p^δ which could be used as an independent generator of G must also contain all the operators of G whose orders divide p^δ. If a characteristic subgroup contains an operator of order p^δ that is found in H_ϵ, it contains also all the operators of $\{H_1, H_2, \ldots, H_\epsilon\}$ whose orders divide p^δ. If G contains one operator of order p^δ which could be used as an independent generator at least $\frac{p-1}{p}$ of its operators of order p^δ could be used separately as independent generators. In any abelian group of order p^a the number of operators of a given order is at least $p-1$ times the number of all those of lower orders. If a subgroup includes more than one-half of the operators of a given order, it must include all the operators of this order and of all lower orders. All the characteristic subgroups, which contain operators of order p^δ, have in common a characteristic subgroup which includes operators of order p^δ. The cyclic group is the only one in which every subgroup is characteristic and the groups of type $(1, 1, 1, \ldots)$ is the only one which has no characteristic subgroup.

If the order of G is not a power of a single prime, the number of its sets of conjugate operators can be more readily stated if the identity is regarded as forming a conjugate set by itself. When this is done, the number of sets of conjugate operators in G is merely the product of the numbers of sets of conjugates in its Sylow subgroups. This product will also be the number of the characteristic subgroups of G (the identity being regarded as a characteristic subgroup) except when the special conditions noted above are satisfied in the Sylow subgroup of order 2^{a_0} contained in G.

Collineations whose Characteristic Determinants have Linear Elementary Divisors with an Application to Quadratic Forms.*

By A. B. Coble.†

Weirstrass' solution of the problem of the equivalence of two non-singular pencils of bilinear forms is based on the simultaneous reduction of two members of a pencil to a normal or canonical form. Segre‡ has employed Weirstrass' normal form to obtain a classification of collineations, and has discussed the projective properties of the various classes.

In general, the reduction to this normal form is not unique. A later method, devised by Darboux|| and perfected by Stickelberger,§ clearly indicates, by the introduction of systems of parameters, a certain freedom in the reduction. However, the conditions to which the parameters are subjected are of a rather vague nature.

In the following only the general class of collineations mentioned above will be treated. A method will be given for writing down in terms of parameters the *most general* normal form of a collineation of this class. Moreover, the restrictions upon the parameters will appear in very simple form.

* A quite complete account of the literature on this subject is given by Muth—Elementartheiler (Teubner, 1899). Only important references will be given hereafter.

† Carnegie Institution, Washington, D. C.

‡ Reale Acad. dei Lincei (1884), Ser. 3a, Vol. XIX, p. 6.

|| Liouville's Journal (1874), Ser. II, Vol. XIX, p. 347.

§ Crelle's Journal (1879), Vol. 86, p. 20.

SECTION I.

§1.—*Generalities on Collineations.**

The bilinear form

$$A \equiv \sum_{\iota, \kappa} a_{\iota\kappa} x_\iota u_\kappa \equiv (ax)(u\alpha) \equiv (bx)(u\beta) \equiv \ldots . \qquad (\iota, \kappa = 1, 2, \ldots . n)$$

in contragredient variables x and u, which are considered as point and plane† coordinates in a linear space of $n-1$ dimensions, S^{n-1}, represents, when equated to zero, two collineations in S^{n-1}, which are inverse. The one collineation is given in the form of a point correspondence, the other in the form of a plane correspondence. As we shall be interested only in points and planes which are unaltered by the collineations,‡ the distinction between the two is not material.

The number and relative position of the fixed points and planes of A is determined by the behavior of the characteristic determinant of A. We assume first that none of the roots of the characteristic equation are zero, i. e. the collineation is non-singular. If any root substituted in the characteristic determinant makes all minors of degree $n-p$ vanish but not all of degree $n-p-1$, there is a linear point spread of dimensions p, P^p, every point of which is a fixed point. Corresponding to this same root, there is a linear plane spread of dimensions p, Π^p, every plane of which is a fixed plane. P^p and Π^p will be called the fixed spreads of the selected root. No point of the fixed spread of one root lies in the fixed point spread of any other root. A point of the fixed spread of one root lies on *every* plane of the fixed plane spreads of *every other* root. A point of the fixed point-spread of a root may or may not lie on all the planes of the fixed plane-spread of that root.

§2.—*Statement of the Problem.*

Collineations of the kind under consideration may be characterized in various ways:

(1). Their characteristic determinants possess only linear elementary divisors.

* See Segre, loc. cit.

† By *plane* is meant a linear spread of dimensions $n-2$ in S^{n-1}.

‡ Such points and planes will be called "fixed."

(2). No point of the fixed point-spread of any root lies on all the planes of the fixed plane-spread of that same root.

(3). If the characteristic determinant has a root of multiplicity $p+1$, that root makes all the minors of the determinant of degree $n-p$ vanish.

Any one of these three properties necessitates the others. It may easily be shown that the fixed point-spaces of the various roots are not only entirely separate but also *linearly independent.*

The given collineation will be said to be in a normal form when it is expressed by means of n fixed points and n fixed planes, which together form a "basis" in S^{n-1}. In the following the most general normal form will be sought, i. e. one which contains parameters, by particularizing which all possible normal forms are obtained. It is further desirable that the geometric meaning of these parameters and of the conditions to which they are subjected, be evident from the reduction; also, that the reduced form itself shall contain explicitly only the ordinary invariant forms of the collineation (including the roots of the characteristic equation).

§3.—*Combined Parametral Representation of the Spaces P^p and Π^p of a Root of Multiplicity $p+1$.*

Let v be a fixed plane of A. Then v satisfies the identity in x,

$$(ax)(va) \equiv \lambda\,(vx),$$

where λ is a root of the characteristic equation of A. Suppose it is a root of multiplicity $p+1$. By hypothesis, it makes all the minors of degree $n-p$ of the characteristic determinant vanish. $p+1$ equations of the identity may be discarded and $n-p-1$ of the coordinates v_i solved for in terms of the others. For symmetry, however, a method employed by Hilbert* for binary forms will be used. The n equations of the identity are extended by the addition of $p+1$ new variables $\rho, \rho_1, \rho_2, \ldots \rho_p$ with arbitrary coefficients. The identity

$$(v_1x_1 + v_2x_2 + \cdots + v_nx_n) - (vx) \equiv 0$$

is added to the system as well as the p relations

$$(vx^1) = 0, \quad (vx^2) = 0, \; \ldots \; (vx^p) = 0.$$

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* Math. Annalen (1887), Vol. 28.

The system then reads

$$
\begin{array}{llllr}
v_1(a_1\alpha_1-\lambda) & +\,v_2a_1\alpha_2 & +\cdots+v_na_1\alpha_n & +\,\rho u_1+\rho_1u_1^1+\cdots+\rho_pu_1^p & =0,\\
v_1a_2\alpha_1 & +\,v_2(a_2\alpha_2-\lambda) & +\cdots+v_na_2\alpha_n & +\,\rho u_2+\rho_1u_2^1+\cdots+\rho_pu_2^p & =0,\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\\
v_1a_n\alpha_1 & +\,v_2a_n\alpha_2 & +\cdots+v_n(a_n\alpha_n-\lambda) & +\,\rho u_n+\rho_1u_n^1+\cdots+\rho_pu_n^p & =0,\\
v_1x_1 & +\,v_2x_2 & +\cdots+v_nx_n & & =0,\\
 & & & & -(vx)\\
v_1x_1^1 & +\,v_2x_2^1 & +\cdots+v_nx_n^1 & & =0,\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\\
v_1x_1^p & +\,v_2x_2^p & +\cdots+v_nx_n^p & & =0,
\end{array}
$$

a system of $n+p+1$ equations in the $n+p+2$ unknowns v_i, ρ, ρ_i, (vx). The values of ρ and ρ_i, as determined from this new system, are zero, since they may be expressed in terms of minors of degree $n-p$. The system as modified above is then equivalent to the original system and the p added relations. Solving for the unknown (vx), we have (the symbol, $\simeq$, being understood to mean "is proportional to"),

$$
(vx)\simeq W_p(\lambda)\equiv(-1)^n\begin{vmatrix}
a_1\alpha_1-\lambda & a_1\alpha_2 & \cdots & a_1\alpha_n & u_1 & u_1^1 & \cdots & u_1^p\\
a_2\alpha_1 & a_2\alpha_2-\lambda & \cdots & a_2\alpha_n & u_2 & u_2^1 & \cdots & u_2^p\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
a_n\alpha_1 & a_n\alpha_2 & \cdots & a_n\alpha_n-\lambda & u_n & u_n^1 & \cdots & u_n^p\\
x_1 & x_2 & \cdots & x_n & 0 & 0 & \cdots & 0\\
x_1^1 & x_2^1 & \cdots & x_n^1 & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
x_1^p & x_2^p & \cdots & x_n^p & 0 & 0 & \cdots & 0
\end{vmatrix}.
$$

Hence $W_p(\lambda)$ is for arbitrary values $x^1, x^2, \ldots x^p$, the equation of the fixed plane of Π^p which passes through the selected points $x^1, \ldots, x^p$ and this regardless of the values of $u, u^1, \ldots, u^p$ if only they are not such as to make $W_p(\lambda)$ vanish identically. But from the self-dual nature of the result, this theorem follows:

(1). *If* λ *is a* $(p+1)$*-tuple root of the characteristic equation,* $W_p(\lambda)$ *is the product of a fixed point of* P^p *and a fixed plane of* Π^p*, the fixed point lying on the selected planes* $u^1, \ldots, u^p$*; the fixed plane passing through the selected points* $x^1, \ldots, x^p$.

The fixed plane is determined rather by the linear space, S^{p-1}, which passes through the selected points $x^1, \ldots, x^p$. Any linearly independent set of p points of this S^{p-1} will determine the fixed plane as well as the selected set.

$W_p(\lambda)$ contains one set of variables x, u and p sets of parameters $x^1, u^1, \ldots, x^p, u^p$. If the parameters do not appear, the determinant will be denoted by $W(\lambda)$. If also the variables do not appear, the (characteristic) determinant will be denoted by $W_0(\lambda)$. Expanding $W_0(\lambda)$ in powers of λ, we may write

$$W_0(\lambda) \equiv \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n.$$

We wish now to obtain the expansions of $W(\lambda)$, $W_1(\lambda), \ldots, W_p(\lambda)$ in terms of c_i, powers of λ and powers of A. For brevity, the symbolic notation for forms will be used. We put

$$E \equiv A^0 \equiv (ux), \quad A^1 \equiv (ax)(u\alpha), \quad A^2 \equiv (ax)(b\alpha)(u\beta), \quad \text{etc.}$$

Consider the series

$$S \equiv \frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \frac{A^2}{\lambda^3} + \ldots$$

which converges for sufficiently large values of λ. Then

$$SA = \frac{A^1}{\lambda} + \frac{A^2}{\lambda^2} + \frac{A^3}{\lambda^3} + \ldots,$$

$$S\lambda E = A^0 + \frac{A^1}{\lambda} + \frac{A^2}{\lambda^2} + \frac{A^3}{\lambda^3} + \ldots,$$

$\therefore$
$$S(\lambda E - A) = A^0 = E$$

and
$$S = (\lambda E - A)^{-1} = \frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \ldots = \frac{W(\lambda)}{W_0(\lambda)},$$

$\therefore$
$$W(\lambda) \equiv (\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n)\left(\frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \frac{A^2}{\lambda^3} + \ldots\right).$$

Since the right-hand member is equivalent to a polynomial, it must itself reduce to a polynomial. Therefore, expanding and neglecting negative powers of λ, we have

$$W(\lambda) \equiv (\lambda^{n-1} + c_1\lambda^{n-2} + \ldots + c_{n-1})A^0$$
$$+ (\lambda^{n-2} + c_1\lambda^{n-3} + \ldots + c_{n-2})A^1 + \ldots + (\lambda + c_1)A^{n-2} + A^{n-1}$$

If $d_{n-\kappa} \equiv \lambda^{n-\kappa} + c_1\lambda^{n-\kappa-1} + \ldots\ldots + c_{n-\kappa}$,

$$W(\lambda) \equiv d_{n-1}A^0 + d_{n-2}A^1 + d_{n-3}A^2 + \ldots\ldots + A^{n-1}.$$

The same result might be obtained by taking the evectant of $W_0(\lambda)$ as to the coefficients of A, and changing the sign. Comparing this result with the one already obtained, we see that the evectant (with negative sign) of c_1 is A^0; of c_2 is $c_1A^0 + A^1$; of c_3 is $c_2A^0 + c_1A^1 + A^2$, etc.

$W_1(\lambda)$ may now be obtained from $W(\lambda)$ by again taking the negative evectant, new variables x^1 and u^1 being introduced. First, it is evident from the form of $W_1(\lambda)$ that all terms in its expansion will fall into one of two sets, the one set being obtained from the other by permuting x and x^1 and changing sign. Also, the terms must be either of the form $\mu . A\,(x, u) . A^{\kappa}(x^1, u^1)$ or of the form $\nu A^{\iota}(x, u^1) . A^{m}(x^1, u)$. The sum of terms of the first form and the sum of terms of the second form make up the two sets above mentioned. In taking the evectant of $W(\lambda)$, terms of the first form arise when A^{ι} is considered constant and c_{ι} variable; terms of the second form when c_{ι} is considered constant and A^{ι} variable. Since the result of evection upon c_{ι} is known, we may write at once

$$W_1(\lambda) \equiv d_{n-2}A^0A^0_{(1)} + d_{n-3}(A^0A^1_{(1)} + A^1A^0_{(1)}) + \ldots\ldots$$
$$+ (A^0A^{n-2}_{(1)} + \ldots\ldots + A^{n-2}A^0_{(1)}) - R_1,$$

where $A^{\iota}_{(1)}$ is A^{ι} written with variables x^1 and u^1 and R_1 is the result of permuting x and x^1 in the previous terms.

By a similar argument, we may write

$$W_2(\lambda) \equiv d_{n-3}A^0A^0_{(1)}A^0_{(2)} + d_{n-4}(A^0A^0_{(1)}A^1_{(2)} + A^0A^1_{(1)}A^0_{(2)} + A^1A^0_{(1)}A^0_{(2)}) + \ldots\ldots \pm R_2,$$

where $\pm R_2$ is the result of permuting x, x^1 and x^2 in the previous terms, the sign of the permutation being attached. In general,

$$W_p(\lambda) \equiv d_{n-p-1}\sum_0 + d_{n-p-2}\sum_1 + \ldots\ldots + \sum_{n-p-1} \pm R_p,$$

where $\sum_{\kappa}$ is the sum of the homogeneous terms of degree κ, composed of powers of A, $A_{(1)}$, $A_{(2)}$, ..., $A_{(p)}$ and R_p, has a meaning entirely analogous to that of R_1 and R_2.

§4.—*The Identical Collineation of the Spaces P^p and Π^p.*

Connected with the series of forms $W_1(\lambda), \ldots, W_p(\lambda)$ is a series of covariant collineations obtained by taking the bilinear invariant as to each pair of parameters x^κ and u^κ, the process being repeated until the parameters no longer appear. Manifestly, the same result will be obtained by taking the successive derivatives of $W(\lambda)$ with respect to λ. We define then

$$W^{(\kappa)}(\lambda) \equiv \frac{1}{\kappa!} \frac{\partial^\kappa}{\partial \lambda^\kappa} W(\lambda).$$

Writing as before

$$W(\lambda) \equiv d_{n-1}A^0 + d_{n-2}A^1 + \ldots + A^{n-1},$$

we may put

$$\begin{aligned}
W^{(1)}(\lambda) &\equiv d^{(1)}_{n-2}A^0 + d^{(1)}_{n-3}A^1 + \ldots + A^{n-2}, \\
W^{(2)}(\lambda) &\equiv d^{(2)}_{n-3}A^0 + d^{(2)}_{n-4}A^1 + \ldots + A^{n-3}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
W^{(p)}(\lambda) &\equiv d^{(p)}_{n-p-1}A^0 + d^{(p)}_{n-p-2}A^1 + \ldots + A^{n-p-1}.
\end{aligned}$$

λ being a $(p+1)$-tuple root of $W_0(\lambda) = 0$, the various sets of coefficients $d^{(k)}$ have perfectly definite values. The d_ι are the coefficients of an equation which has all the roots of $W_0(\lambda) = 0$ except *one* root λ. The $d^{(1)}_\iota$ are the coefficients of an equation which has all the roots of $W_0(\lambda) = 0$ except two roots λ, etc. Finally the $d^{(p)}_\iota$ are the coefficients of an equation which has all the roots of $W_0(\lambda)$ except $p+1$ roots λ. Hence, among the sets of coefficients, we have the relations

$$d^{(\rho)}_\kappa = d^{(\rho+1)}_\kappa - \lambda d^{(\rho+1)}_{\kappa-1},$$
$$(\rho = 0, 1, \ldots, p-1)(\kappa = 1, 2, \ldots, n-\rho).$$

Of these collineations $W(\lambda)$, $W^{(1)}(\lambda)$, $\ldots$, $W^{(p-1)}(\lambda)$ vanish identically, for they are expressible in terms of minors of degree $n-p$ or greater. With regard to $W^{(p)}(\lambda)$, we have the theorem:

(2). $W^{(p)}(\lambda)$ *is a singular collineation. Its singular point-spread is the linear spread determined by the fixed point-spreads of all the roots other than* λ.

For, let x be a fixed point of A corresponding to the root λ_1. Then the homolog x by $W^{(p)}(\lambda)$ is

$$(d^{(p)}_{n-p-1}+\lambda_1 d^{(p)}_{n-p-2}+\lambda_1^2 d^{(p)}_{n-p-3}+\ldots\ldots+\lambda_1^{n-p-1})(ux)=0.$$

Since λ_1 is a root of the equation with coefficients $d^{(p)}_i$, the first factor vanishes, i. e. x is a singular point of $W^{(p)}(\lambda)$. Also, it is evident that—

(3). *For the fixed point-spread of the root* λ, $W^{(p)}(\lambda)$ *is the identical collineation.*

Since every point is linearly expressible in terms of all the fixed point-spreads of A, it follows that—

(4). *If the linear spread of dimensions* $n-p-1$ *determined by the fixed point-spreads of all the roots other than* λ, *and a point* x, *cuts* P^p *in a point* x', *then* x' *is the homolog of* x *by* $W^{(p)}(\lambda)$.

$W^{(p)}(\lambda)$ projects, therefore, a point, in general position, upon the fixed spread P^p. Analytically, $W^{(p)}(\lambda)$ may be employed as a transformation which replaces a general point x by a fixed point of P^p. Properties of $W^{(p)}(\lambda)$ dualistic to those stated, may at once be supplied.

§5.—*Transformation of the Determinant* $W_p(\lambda)$.

The parameters $x^1, \ldots, x^p$, appearing in $W_p(\lambda)$, have thus far been unrestricted. Instead of selecting these p points arbitrarily, we will now require that they be p points of the spread P^p. $W_p(\lambda)$ will still represent *any* plane of Π^p. For a plane of Π^p cuts P^p in an S^{p-1}, and the p arbitrary points may be selected in this S^{p-1}. This new choice of parameters may e b analytically effected by replacing the variables x^κ by new variables x^κ by means of $W^{(p)}(\lambda)$ according to theorem (4). The effect of this transformation will be followed out in the expanded form of $W_p(\lambda)$.

We suppose, first, that $W_0(\lambda)=0$ has, besides the root λ of multiplicity $p+1$, the further roots $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ of multiplicity $p_1+1, p_2+1, \ldots, p_\sigma+1$ respectively. Then

$$\begin{aligned}
W(\lambda) &\equiv (A-\lambda A^0)^p (A-\lambda_1 A^0)^{p_1+1}\ldots(A-\lambda_\sigma A^0)^{p_\sigma+1} \equiv (A-\lambda A_0)^p f(A),\\
W^{(1)}(\lambda) &\equiv (A-\lambda A^0)^{p-1} f(A),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
W^{(p)}(\lambda) &\equiv f(A).
\end{aligned}$$

We employ now the following theorem due to Frobenius:*

(5). *If $\psi(\mu)$ is the n^{th} elementary divisor of the characteristic determinant of a bilinear form of 2n variables, then $\psi(A) = 0$ is the equation of lowest degree which the form A satisfies.*

Since, in the present case, the elementary divisors are all linear, we have, as the identity of lowest degree which the form A satisfies:

$$(A - \lambda A^0)(A - \lambda_1 A^0)(A - \lambda_2 A^0) \ldots\ldots (A - \lambda_\sigma A^0) \equiv 0 .$$

Recalling the value of $f(A)$, we have, as a consequence of this identity:

$$W^{(p)}(\lambda) . (A - \lambda A^0) \equiv 0,$$

or

$$W^{(p)}(\lambda) . A \equiv \lambda W^{(p)}(\lambda),$$

hence

$$W^{(p)}(\lambda) . A^2 \equiv \lambda^2 W^{(p)}(\lambda), \text{ etc.}$$

Arranging the expanded form of $W_p(\lambda)$ in powers of $A_{(p)}$, it takes the form

$$A^0_{(p)}\left[d_{n-p-1}\sum_0 + d_{n-p-2}\sum_1 + \ldots\ldots + \sum_{n-p-1}\right] \\ + A^1_{(p)}\left[d_{n-p-2}\sum_0 + d_{n-p-3}\sum_1 + \ldots\ldots + \sum_{n-p-2}\right] + \ldots\ldots \\ + A^{n-p-2}_{(p)}\left[d_1\sum_0 + \sum_1\right] + A^{n-p-1}_{(p)}\sum_0 \pm R_p .$$

We note at once that in operating on this expression, R_p may be neglected, the result of operation on its term being inferred from the result on the previous terms. Also, where no ambiguity may occur, we write $W^{(p)}(\lambda)$ as $W^{(p)}$.

We replace now the variables x^p in $A^i_{(p)}$ by operating with variables u in $W^{(p)}(u, x^p)$. According to the above identities, the result is simply $\lambda^i W^{(p)}(x^p, u^p)$. $W^{(p)}(x^p, u^p)$ will then factor out from the above expression. The remaining factor is

$$\left[d_{n-p-1}\sum_0 + \ldots\ldots + \sum_{n-p-1}\right] + \lambda\left[d_{n-p-2}\sum_0 + \ldots\ldots + \sum_{n-p-2}\right] + \ldots\ldots \\ + \lambda^{n-p-2}\left[d_1\sum_0 + \sum_1\right] + \lambda^{n-p-1}\sum_0$$

* Crelle's Journal (1878), Vol. 84, p. 12.

in which the coefficient of $\sum\limits_{\iota-1}$ is

$$d_{n-p-\iota} + \lambda d_{n-p-\iota-1} + \ldots\ldots \lambda^{n-p-} = d^{(1)}_{n-p-\iota},$$

since $d_\kappa = d^{(1)}_\kappa - \lambda d^{(1)}_{\kappa-1}$.

The factor in question is then

$$d^{(1)}_{n-p-1}\sum_0 + d^{(1)}_{n-p-2}\sum_1 + \ldots\ldots + d^{(1)}_1\sum_{n-p-2} + \sum_{n-p-1}.$$

Proceeding in the same way with this factor, we find, on transforming the parameters x^{p-1}, that $W^{(p)}(x^{p-1}, u^{p-1})$ separates, leaving a new factor

$$d^{(2)}_{n-p-1}\sum_0 + \ldots\ldots + \sum_{n-p-1}.$$

Finally, we have, as a last factor after transformation of all the parameters

$$d^{(p)}_{n-p-1}A^0 + d^{(p)}_{n-p-2}A^1 + \ldots\ldots + A^{n-p-1} \equiv W^{(p)}(x, u).$$

The result, then, of the transformation of the parameters in the principal part of $W_p(\lambda)$ is

$$W^{(p)}(x, u) \,.\, W^{(p)}(x^1, u^1) \,.\, W^{(p)}(x^2, u^2) \ldots\ldots W^{(p)}(x^p, u^p).$$

To obtain the complete result, we permute $x, x^1, \ldots\ldots, x^p$ in all possible way, prefix the sign of the permutation and add. This result may be expressed thus:

(6). *If in $W_p(\lambda)$ the parameters $x^1, \ldots\ldots, x^p$ are transformed by means of $W^{(p)}(\lambda)$, the result may be put in the form*

$$V_p(\lambda) \equiv \begin{vmatrix} W^{(p)}(x, u) & W^{(p)}(x^1, u) & \ldots\ldots & W^{(p)}(x^p, u), \\ W^{(p)}(x, u^1) & W^{(p)}(x^1, u^1) & \ldots\ldots & W^{(p)}(x^p, u^1), \\ \ldots & \ldots & \ldots & \ldots \\ W^{(p)}(x, u^p) & W^{(p)}(x^1, u^p) & \ldots\ldots & W^{(p)}(x^p, u^p), \end{vmatrix}.$$

§6.—*The Normal Form of $W^{(p)}(\lambda)$.*

In the identity of theorem 6 we assume

$$W^{(p)}(x^\iota, u^\kappa) = 0, \ \iota \neq \kappa \text{ and } W^{(p)}(x^\iota, u^\iota) \neq 0.$$

Let $\prod_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ denote the product of the p factors $W^{(p)}(x^1, u^1)$, $W^{(p)}(x^2, u^2)$ .... $W^{(p)}(x^p, u^p)$ and let $\prod_1^p{}_{\iota\kappa} W^{(p)}(x^\iota, u^\iota)$ denote the result of replacing in $\prod_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ the factor $W^{(p)}(x^\kappa, u^\kappa)$ by $W^{(p)}(x, u^\kappa) \,.\, W^{(p)}(x^\kappa, u)$, then we may expand the determinant for $V_p(\lambda)$ in the form

$$V_p(\lambda) = W^{(p)}(x, u) \prod_1^p{}_\iota W^{(p)}(x^\iota, u^\iota) - \sum_{\kappa=1}^p \prod^p{}_{\iota\kappa} W^{(p)}(x^\iota, u^\iota).$$

Solving this equation for $W^{(p)}(x, u) \equiv W^{(p)}(\lambda)$, we have

$$\text{I.} \qquad W^{(p)}(\lambda) \equiv \frac{V_p(\lambda) + \sum\limits_{\kappa=1}^p \prod_{\iota\kappa} W^{(p)}(x^\iota, u^\iota)}{\prod\limits_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)}.$$

$W^{(p)}(\lambda)$ is then expressed in terms of the p points $W^{(p)}(x^\iota, u)$ and the p planes $W^{(p)}(x, u^\iota)$ $(\iota = 1, 2, \ldots, p)$ and the point and plane $V_p(\lambda)$. Each of the p points lies on all the p planes except that which has the same index ι. For the bilinear invariant of $W^{(p)}(x^\iota, u)$ and $W^{(p)}(x, u^\kappa)$ is $[W^{(p)}(x^\iota, u^\kappa)]^2$ or $f(\lambda) \,.\, W^{(p)}(x^\iota, u^\kappa)$. Since $f(\lambda) \neq 0$, this does or does not vanish according as ι is not or is equal to κ. Also, from the above property of the square of $W^{(p)}(\lambda)$, it follows that the result of operating with $W^{(p)}(x^\iota, u)$ or $W^{(p)}(x, u^\iota)$ on $V_p(\lambda)$ vanishes.

We still have to show that under the conditions put upon the parameters, the point and plane given by $V_p(\lambda)$ are not incident, i. e. that the bilinear invariant of $V_p(\lambda)$ is not zero. The bilinear invariant of the p negative terms of V_p is $-p \,.\, f(\lambda) \,.\, \prod_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$; of the positive term is $\prod_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ times the bilinear invariant of $W^{(p)}$. Since the bilinear invariant of A^κ is $\Sigma\lambda_p^\kappa$, the summation, including all roots, we may write the bilinear invariant of $W^{(p)}(\lambda)$ in the form

$$(p+1)f(\lambda)+(p_1+1)f(\lambda_1)+\ldots\ldots+(p_\sigma+1)f(\lambda_\sigma),$$

of which all terms vanish but the first. Hence the bilinear invariant of $V_p(\lambda)$ is

$$\{(p+1)-p\}f(\lambda).\prod_{1}^{p}{}_{\iota}\, W^{(p)}(x^\iota, u^\iota)\neq 0,$$

From this follows the theorem:

(7). *The formula* I *gives the most general normal form of* $W^{(p)}(\lambda)$, *the identical collineation in the spaces* P^p *and* Π^p. *The points and planes occuring in the formula are situated in* P^p *and* Π^p *respectively. Points and planes with different indices are incident (the point and plane,* V_p, *being supposed to have the same index). The parameters are subjected only to the conditions*

$$W^{(p)}(x^\iota, u^\varkappa)=0,\ \iota\neq\varkappa \quad \textit{and} \quad W^{(p)}(x^\iota, u^\iota)\neq 0, \qquad (\iota, \varkappa=1, 2, \ldots\ldots, p).$$

§7.—*The Normal Form of the Collineation A.*

Denoting by $f_\iota(\mu)$ a polynomial which has all the roots of $W_0(\mu)$ except the $p_\iota+1$ roots λ_ι, then $f_\iota(A)\equiv W^{(p_\iota)}(\lambda_\iota)$. A treatment entirely similar to that of $W^{(p)}(\lambda)$ will give for each $W_\iota^{(p)}(\lambda_\iota)$ the corresponding normal form. Also put

$$\phi(A)\equiv(A-\lambda A^0)(A-\lambda_1 A^0)\ldots\ldots(A-\lambda_\sigma A^0)\equiv 0,$$

$f_\iota(A)$ is a polynomial in A. The highest degree to which A may occur is $n-1$ (in case λ_ι is a simple root). Suppose the highest exponent of A in $f(A)$ or $f_\iota(A)$ is $\varkappa$. To the $\sigma+1$ identities

$$f_\iota(A)\equiv W^{(p_\iota)}(\lambda_\iota),$$

which may be viewed as equations in the unknown $A^0, A^1, A^2, \ldots\ldots, A^\varkappa$, we add that the $\varkappa-\sigma$ identities

$$A^\iota\phi(A)\equiv 0, \qquad \iota=0, 1, \ldots\ldots, \varkappa-\sigma-1.$$

The determinant of this system of $\varkappa+1$ non-homogeneous equations in the $\varkappa+1$ unknowns $A^0, \ldots\ldots, A^\varkappa$ is not zero. For the vanishing of this determinant

would require the existence of quantities $\rho, \rho_1, \ldots, \rho_\kappa$, not all zero, and such that

$$\rho f(\mu) + \rho_1 f_1(\mu) + \ldots + \rho_\sigma f_\sigma(\mu) + \rho_{\sigma+1}\phi(\mu) + \ldots + \rho_\kappa \mu^{\kappa-\sigma-1}\phi(\mu) \equiv 0$$

for all values of μ. But for $\mu = \lambda$, the identity becomes simply $\rho f(\lambda) = 0$ or $\rho = 0$. Similarly, we show that $\rho_\iota = 0$ ($\iota = 1, 2, \ldots, \sigma$). The identity must then take the form

$$\rho_{\sigma+1}\phi(\mu) + \rho_{\sigma+2}\mu\phi(\mu) + \ldots + \rho_\kappa \mu^{\kappa-\sigma-1}\phi(\mu) \equiv 0,$$

which is valid only if $\phi(\mu) \equiv 0$. Hence the determinant of the above system does not vanish. The system may then be solved for the unknown A^1, and we have

II. $$A^1 \equiv b W^{(p)}(\lambda) + b_1 W^{(p_1)}(\lambda_1) + \ldots + b_\sigma W^{(p_\sigma)}(\lambda_\sigma),$$

where the b_ι are functions of the roots of the characteristic equation only.

We have then the theorem:

(A). *The formula* II, *in which* $W^{(p_\iota)}(\lambda_\iota)$ *is replaced by its normal form according to formula* I, *is the required general normal form of the collineation* A.

That the n points and planes of this normal form make up a "basis" in S^{n-1} is at once seen from the fact that they are all fixed points and planes and that every fixed point (plane) of one root is incident with every fixed plane (point) of every other root.

The same set of equations gives as well the values of certain powers of A, and in particular of A^0. Solving for A^0, we have

III. $$A^0 = b' W^{(p)}(\lambda) + b_1' W^{(p_1)}(\lambda_1) + \ldots + b_\sigma' W^{(p_\sigma)}(\lambda_\sigma).$$

From formulæ II and III theorem (B) follows:

(B). *The identical collineation in* S^{n-1} *may be expressed in terms of the points and planes which enter into the general normal form of the collineation* A.

In a way this is trivial, since those points and planes form a "basis" in S^{n-1}.

But by the use of theorem (B) we may obtain the normal form of A as given

by the usual methods. We will show first that

$$b = \lambda b' \quad \text{and} \quad b_\iota = \lambda_\iota b'_\iota, \qquad \iota = 1, 2, \ldots, \sigma.$$

Since $\qquad f(A) \equiv W^{(p)}(\lambda),$ then $Af(A) \equiv \lambda W^{(p)}(\lambda).$

Hence the $\sigma + 1$ identities

$$Af_\iota(A) \equiv \lambda_\iota W^{(p_\iota)}(\lambda_\iota),$$

and the $\varkappa - \sigma$ identities

$$A^{\iota+1}\phi(A) \equiv 0 \qquad (\iota = 0, 1, \ldots, \varkappa - \sigma - 1)$$

form the same system to obtain A as the original system to obtain A^0, except for the additional factor λ_ι in the right-hand members.

$$\therefore \qquad A^1 \equiv b'\lambda W^{(p)}(\lambda) + b'_1\lambda_1 W^{(p_1)}(\lambda_1) + \ldots + b'_\sigma \lambda_\sigma W^{(p_\sigma)}(\lambda_\sigma).$$

A comparison of this with formula II verifies the above assertion. There follows then

(C). *If the identical collineation in the form* III *be written again as*

$$A^0 \equiv U_1X_1 + U_2X_2 + \ldots + U_nX_n,$$

the collineation A in the form II *will take the new form*

$$\begin{aligned} A \equiv \lambda\,(U_1X_1 + \ldots + U_{p+1}X_{p+1}) \\ + \lambda_1(U_{p+2}X_{p+2} + \ldots + U_{p+p_1+2}X_{p+p_1+2}) + \ldots \\ + \lambda_\sigma(U_{n-p_\sigma}X_{n-p_\sigma} + \ldots + U_nX_n). \end{aligned}$$

This is the normal form as usually written.

The peculiar relations of the minors of the determinant of the above system are worth notice, though easily read off from the above development.

For later use, we consider further the case of $p = n - 1$, i. e. the case of A^1, the identical collineation in S^{n-1}. $W^{(p)}(\lambda)$ vanishes identically; $W_p(\lambda)$ becomes simply the determinant product

$$|x, x^1, x^2, \ldots, x^p|\,|u, u^1, u^2, \ldots, u^p|.$$

The transformation (§5) of $W_p(\lambda)$ into the determinant form $V_p(\lambda)$ here

becomes simply the multiplication of the determinant and our fundamental formula becomes

$$|x, x^1, x^2, \ldots, x^p|\,|u, u^1, u^2, \ldots, u^p| \equiv \begin{vmatrix} (ux) & (ux^1) & \ldots & (ux^p) \\ (u^1x) & (u^1x^1) & \ldots & (u^1x^p) \\ \ldots & \ldots & \ldots & \ldots \\ (u^px) & (u^p, x^1) & \ldots & (u^p, x^p) \end{vmatrix}.$$

Assuming as before $(u_\iota x_\kappa) = 0$, $\iota \neq \kappa$ and $(u^\iota x^\iota) \neq 0$, also writing $\prod_1^p{}_\iota (u_\iota x_\iota)$ for $(u^1 x^1)(u^2 x^2) \ldots (u^p x^p)$, but

$$\prod_1^p{}_{\iota\kappa} (u^\iota x^\iota) \text{ for } (u^1 x^1) \ldots (u^{\kappa-1} x^{\kappa-1})(ux^\kappa)(u^\kappa x)(u^{\kappa+1} x^{\kappa+1}) \ldots (u^p x^p),$$

we may solve for (ux), obtaining

$$(ux) \equiv \frac{|x, x^1, \ldots, x^p|\,|u, u^1, \ldots, u^p| + \sum_{\kappa=1}^{p} \prod_1^p{}_{\iota\kappa} (u^\iota x^\iota)}{\prod_1^p{}_\iota (u^\iota x^\iota)}$$

This is the expression of the identical collineation (ux) in S^{n-1} in terms of the most general basis of S^{n-1}.

Section II.

The Reduction to Normal Form of Non-singular Pencils of Quadratic Forms whose Determinants have only Linear Elementary Divisors.

§1.—*The Collineation Determined by the Groundforms.*

Let

$$(ax)^2 \equiv \sum_{\iota, \kappa} a_{\iota\kappa}\, x_\iota\, x_\kappa$$

and

$$(bx)^2 \equiv \sum_{\iota, \kappa} b_{\iota\kappa}\, x_\iota\, x_\kappa,$$

where

$$a_{\iota\kappa} = a_{\kappa\iota} \text{ and } b_{\iota\kappa} = b_{\kappa\iota}, \qquad \iota, \kappa = 1, 2, \ldots, n,$$

be two quadratic forms such that the determinant of $(ax)^2 - \lambda (bx)^2$ has only linear elementary divisors. We may suppose also

$$|a_{\iota\kappa}| \neq 0 \quad \text{and} \quad |b_{\iota\kappa}| \neq 0.$$

For, if the pencil of forms is non-singular, two non-singular members may be selected as groundforms. Let

$$(u\beta)^2 \equiv \sum_{\iota, \kappa} \beta_{\iota\kappa} u_\iota u_\kappa \qquad \iota, \kappa = 1, 2, \ldots, n$$

be the reciprocal to $(bx)^2$, so that

$$(u\beta)(b\beta)(bx)^2 \equiv (ux).$$

The two groundforms equated to zero represent in S^{n-1} two quadratic surfaces, $(u\beta)^2 = 0$ being the quadratic $(bx)^2 = 0$ in reciprocal coordinates.

Consider now the collineation

$$A \equiv (ax)(a\beta)(u\beta) = 0,$$

which is set up by taking for the homolog of x the polar point as to $(u\beta)^2 = 0$ or $(bx)^2 = 0$ of the polar plane of x as to $(ax)^2 = 0$.

(1). *The characteristic determinant of the collineation A has also only linear elementary divisors.*

For the determinant of the pencil of quadratic forms is that of the pencil of bilinear forms

$$(ax)(ay) - \lambda (bx)(by). \tag{2}$$

Transform the variables y by operating upon them with the variables v of $(u\beta)(v\beta)$ and there is obtained the new pencil of bilinear forms

$$\left.\begin{array}{l} (u\beta)(ax)(a\beta) - \lambda (b\beta)(bx)(u\beta) \\ (ax)(a\beta)(u\beta) - \lambda (ux). \end{array}\right\} \tag{3}$$

or

Since the pencil (2) may be transformed into ("is equivalent to") the pencil (3), according to a theorem of the theory of forms, the elementary divisors of the pencil (3) coincide with those of the pencil (2).

We suppose, as before, that the characteristic equation of A has $\sigma + 1$ roots $\lambda, \lambda_1, \ldots, \lambda_\sigma$ of multiplicities $p+1, p_1+1, \ldots, p_\sigma + 1$ respectively. Also, we retain for the collineation A and its allied forms the notation of Sec. I.

From the definition of the collineation A and of its fixed planes and points, this theorem follows:

To every fixed point of P^p is coordinated a fixed plane of Π^p, the point and plane being pole and polar as to both $(ax)^2 = 0$ and $(bx)^2 = 0$ or $(u\beta)^2 = 0$. The fixed point and plane will be called "corresponding."

It is this correspondence which differentiates the collineation of the form A from the more general or rather the more arbitrary collineation treated in Sec. I.

§2.—*A Further Transformation of Parameters.*

We have already obtained an expression of the identical collineation A^0* (in the space S^{n-1} to which A refers) in terms of the fixed points and planes of the various roots. For our present purpose a further specialization of the parameters is necessary. We seek to dispose of the parameters which enter into the expression of A^0 so that point and plane of the same index will be *corresponding.*

First the point y and the plane v whose product is given by $W_p(\lambda)$ are such that y lies on the arbitrary planes $u^1, u^2, \ldots, u^p$ and v passes through the arbitrary points $x^1, x^2, \ldots, x^p$. This set of points being chosen we may take u^1 to be the polar plane of x^1, u^2 the polar plane of x^2, etc., all as to $(ax)^2 = 0$. Since y lies on $u^1, u^2, \ldots u^p$ its corresponding plane passes through $x^1, x^2, \ldots x^p$ and being a fixed plane must be v. Hence:

If in $W_p(\lambda)$, u^ι_κ $(\iota = 1, 2, \ldots p;\ \kappa = 1, 2, \ldots n)$ is replaced by $(ax^\iota)a_\kappa$, $W_p(\lambda)$ is the product of corresponding point and plane.

This holds whether or not the parameters refer to fixed points, i.e. this transformation of the parameters U^ι_κ may also be carried out on $V_p(\lambda)$ and the result will be the product of corresponding point and plane. The expression of $V_p(\lambda)$ in determinant form in both variables and parameters and may be written

* Sec. I, §7, Theorem (B).

$$[V_{(p)}(\lambda)]_{u^\iota_\kappa=(ax^\iota)a_\kappa} = \left| \begin{array}{c} W^{(p)}(x, u) \ldots . W^{(p)}(x^p, u) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ W^{(p)}(x, u^p) \ldots . W^{(p)}(x^{(p)}, u^{(p)}) \end{array} \right|_{u^\iota_\kappa=(ax^\iota)a_\kappa} \tag{6}$$

$$\iota = 1, 2, \ldots . p, \qquad \kappa = 1, 2, \ldots . n.$$

As before we assume

$$\left. \begin{array}{l} [W^{(p)}(x^l, u^m)]_{u^\iota_\kappa=(ax^\iota)a_\kappa} = 0 \text{ for } l \neq m, \\ [W^{(p)}(x^l, u^l)]_{u^\iota_\kappa=(ax^\iota)a_\kappa} \neq 0, \qquad (l, m = 1, 2, \ldots . p). \end{array} \right\} \tag{7}$$

And, as before, we may solve for $W^{(p)}(x, u)$.

There remains to show that the other points and planes of the same index, e. g. $W^{(p)}(x^s, u)$ and $[W^{(p)}(x, u^s)_{u^s_\kappa=(ax^s)a_\kappa}$ are also corresponding. Now the polar of $W^{(p)}(x^s, u)$ as to $(ax)^2 = 0$ is $[W^{(p)}(x^s, u)]_{u_\kappa=(ax)a_\kappa}$. $W^{(p)}(x^s, u)$ may be expressed in powers of A, every power of A in which x is replaced by x^s and u_κ by $(ax)\,a_\kappa$ is symmetrical in x^s and x. Hence

$$[W^{(p)}(x^s, u)]_{u_\kappa=(ax)a_\kappa} \equiv [W^{(p)}(x, u^s)]_{u^s_\kappa=(ax^s)a_\kappa}, \tag{8}$$

and all points and planes of the same index are corresponding. We have finally then

(9). *The formula I of* §7, *Sec. I, in which the parameters* u^ι_κ *are replaced by* $(ax^\iota)\,a_\kappa$, *gives the most general normal form of* $W^{(p)}(\lambda)$ *for which point and plane of the same index are corresponding. The restrictions upon the parameters are given by* (7).

We will denote $W^{(p)}(\lambda)$ when the parameters have been disposed as described in the theorem by $[W^{(p)}(\lambda)]'$.

§3.—*The Simultaneous Reduction of* $(ax)^2$ *and* $(bx)^2$ *to Sums of Squares.*

We may proceed as in the last paragraph with the other roots and obtain the most general expression of $W^{(p_\iota)}(\lambda_\iota)$ in the form $[W^{(p_\iota)}(\lambda_\iota)]'$. Using the primed forms, we have from theorem (B), §8, Sec. I:

The identical collineation A^0 in S^{n-1} may be put in the form

$$(ux) \equiv b'[W^{(p)}(\lambda)]' + b'_1[W^{(p_1)}(\lambda_1)]' + \cdots\cdots + b'_\sigma[W^{(p_\sigma)}(\lambda_\sigma)]', \qquad (10)$$

in which every point and plane of the same index are corresponding.

Formula (10) is an identity. We may replace on both sides then u_κ by $(ax)a_\kappa$. The left-hand member becomes $(ax)^2$. The right-hand member, as it stands, is a sum of terms of the form $p_\iota l_\iota$, where the fixed plane l_ι is the polar of the fixed point p_ι as to $(ax)^2 = 0$. To replace u_κ by $(ax)a_\kappa$ is to change p_ι into l_ι. The right-hand member reduces then to a sum of square Σl_ι^2. Let us call the result of this substitution for u in $[W^{(p_\iota)}(\lambda_\iota)]'$, $[W^{(p_\iota)}(\lambda_\iota)]''$. Then is

$$(ax)^2 \equiv \sum_\iota [b'_\iota W^{(p_\iota)}(\lambda_\iota)]'', \qquad \iota = 0, 1, \ldots\ldots, \sigma.$$

We may also replace in the identity u_κ by $(bx)b_\kappa$. On the left-hand side $(bx)^2$ arises. On the right-hand side we obtain terms of the form $l_\iota[p_\iota]_{u_\kappa=(bx)b_\kappa}$. Since the polar of p_ι as to $(ax)^2 = 0$ is l_ι and the polar of l_ι as to $(u\beta)^2 = 0$ is λp_ι if a term of the root λ is in question, the polar of λp_ι as to $(u\beta)^2 = 0$ is l_ι. Hence,

$$[p_\iota]_{u_\kappa=(bx)b_\kappa} = \frac{1}{\lambda}[p_\iota]_{u_\kappa=(ax)a_\kappa}. \qquad (11)$$

Replacing u_κ by $(bx)b_\kappa$ on the left and by $(ax)a_\kappa$ on the right, introducing at the same time the factors $\frac{1}{\lambda_\iota}$ to preserve the identity, we have

$$(bx)^2 = \sum_\iota \frac{1}{\lambda_\iota}[b'_\iota W^{(p_\iota)}(\lambda_\iota)]'', \qquad \iota = 0, 1, 2, \ldots\ldots, \sigma.$$

The preceding results may finally be incorporated in the theorem :

(A). *If the formula* III *of* §7, *Sec. I, be applied to the collineation A and $W^{(p_\iota)}(\lambda_\iota)$, in which the variables u_κ are replaced by $(ax)a_\kappa$ and every set of parameters u^i_κ are replaced by $(ax^i)a_\kappa$, be denoted by $[W^{(p_\iota)}(\lambda_\iota)]''$, then the formulae*

and
$$\left.\begin{aligned}(ax)^2 &\equiv \sum_{\iota} b'_{\iota}[W^{(p_{\iota})}(\lambda_{\iota})]'',\\ (bx)^2 &\equiv \sum_{\iota} \frac{1}{\lambda_{\iota}} b'_{\iota}[W^{(p_{\iota})}(\lambda_{\iota})]''\end{aligned}\right\} \tag{12}$$

give the most general simultaneous reduction of the two quadratic forms to sums of squares. The parameters are subjected only to the conditions

$$[W^{(p_{\iota})}(x^l, u^m)]_{u^m_{\varkappa}=(ax^m)a_{\varkappa}} \begin{cases} = 0 \text{ if } l \neq m \\ \neq 0 \text{ if } l = m \end{cases} \left\{\begin{aligned} \iota &= 0,\ 1,\ 2,\ \ldots,\ \sigma, \\ \varkappa &= \quad 1,\ 2,\ \ldots,\ n, \\ l,\ m &= \quad 1,\ 2,\ \ldots,\ p_{\iota}. \end{aligned}\right\} \tag{13}$$

§4.—*Related Geometric Facts.*

Two quadrics such that the pencil of quadratic forms determined by them is of the sort treated, possess geometric relations of considerable interest. For example, our present case may be characterized as follows:

(14). *If the two quadrics in* S^{n-1}, $(ax)^2 = 0$ *and* $(bx)^2 = 0$, *have contact at any point, they have contact on a quadric (in general of dimensions less than* $n-2$) *which contains the point.*

First, if all the roots of the characteristic equation are distinct, there is no contact, for contact at a point requires the incidence of corresponding fixed point and plane, and, therefore, the existence of a double root at least. But for a root of multiplicity $p+1$, the linear spread P^p cuts either quadric in a quadric of dimension $p-1$, say Q^{p-1}. But, by definition, if a fixed point lies on one quadric, it lies on both, and they touch at this point. The two groundforms will then touch along Q^{p-1}.

(15). *The quadric* $[W^{(p_{\iota})}(\lambda_{\iota})]_{u_{\varkappa}=(ax)a_{\varkappa}} = 0$ *has for double points all points of the linear spread determined by the fixed spreads of the roots other than* λ_{ι}. *It touches* $(ax)^2 = 0$ *and* $(bx)^2 = 0$ *along the quadric of contact* $Q^{p_{\iota}-1}$.

The polar system of the quadric

$$[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ax)\,a_\kappa}=0 \tag{16}$$

is given by

$$[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ay)\,a_\kappa}=0, \tag{17}$$

for, according to (8), the form (17) is symmetrical in x and y. This bilinear form (17) gives for a point x a plane which is the polar plane as to $(ax)^2=0$ of the homolog of x by $W^{(p_\iota)}(\lambda_\iota)$. Hence, every singular point of $W^{(p_\iota)}(\lambda_\iota)$ is a double point of the quadric (15). Since $(ax)^2=0$ is non-singular, no other singular points exist. Also, if x is a point of $Q^{p_\iota-1}$, its homolog by $W^{(p_\iota)}(\lambda_\iota)$ is again x. The polar as to $(ax)^2=0$ is then the tangent plane of (15) and $(ax)^2=0$.

The quadric (15) contains the whole linear spread determined by a point of $Q^{p_\iota-1}$ and all the fixed point-spreads of the roots other than λ_ι. Since the tangent planes to $(ax)^2=0$ at points of $Q^{p_\iota-1}$ also pass through the same fixed spreads and are the only tangent planes which do so, we may say:

(18). *The quadric* $[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ax)\,a_\kappa}=0$ *is the envelope of the common tangent planes of* $(ax)^2=0$ *and* $(bx)^2=0$ *at their points of contact on* $Q^{p_\iota-1}$.

The conditions upon the parameters in the reduction of theorem (A), §3, have now a simple geometric meaning. Speaking only of the parameters entering in $[W^{(p_\iota)}(\lambda_\iota)]''$, first x^1 may be chosen at random; x^2 at random upon the polar plane of x^1 as to the quadric (15); x^3 at random upon the polar planes of both x^1 and x^2 as to (15), etc. Finally, x^{p_ι} is chosen at random upon the polar planes of $x^1, x^2, \ldots, x^{p_\iota-1}$ as to (15).

§5.—*Reduction of a Single Quadratic Form to a Sum of Squares.*

We have obtained in §7, Sec. I, when A itself reduces to the identical collineation, an expression of A in terms of the most general basis in S^{n-1}. Under our present hypotheses, A becomes the identical collineation only when $(ax)^2=0$ and $(bx)^2=0$ coincide. Supposing then $(ax)^2\equiv(bx)^2$, we put the resulting collineation in its most general normal form, and again replace u_κ by $(ax)\,a_\kappa$ and

u_κ^ι by $(ax^\iota)\,a_\kappa$. We have then the most general expression of $(ax)^2=0$ as a sum of the squares of planes. The condition upon the parameters are

$$(ax^\iota)(ax^\kappa)\left\{\begin{array}{l}=0 \text{ for } \iota\neq\kappa,\\ \neq 0 \text{ for } \iota=\kappa,\end{array}\right\} \qquad \iota,\kappa=1,\,2,\,\ldots,\,n-1.$$

This is, of course, only the well-known expression of a quadric in S^{n-1} by means of a "self-polar" basis.

BALTIMORE, *Nov.* 1908.

Concerning Certain Elliptic Modular Functions of Square Rank.

By John A. Miller.

Introduction.

In his "Vorlesungen* über die Theorie der Elliptischen Modulfunctionen," Professor Felix Klein has defined the functions†

$$X_{\frac{\alpha}{n}}(u \mid \omega_1, \omega_2) = C_\alpha \prod_{\mu=0}^{n-1} \sigma_{\frac{\alpha}{n}+\epsilon, \frac{\mu+}{n}}(u \mid \omega_1, \omega_2), \tag{1}$$

where u is the fundamental variable of the elliptic function; ω_1, ω_2, are the periods of the elliptic integral of the first kind; $\varepsilon = 0$ or $\frac{1}{2}$ according as n is odd or even, C_α is a quantity independent of u, and

$$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(u \mid \omega_1, \omega_1) = e^{\frac{\lambda\eta_1 + \mu\eta_2}{n}\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)} \sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{n} \,\middle|\, \omega_1, \omega_2\right), \tag{2}$$

where $\sigma(u \mid \omega_1, \omega_2)$ is the ordinary Weierstrassian sigma-function; η_1, η_2, are the periods of elliptic integrals of the second kind and α, n, λ, μ are integers. He has shown that these functions are of rank (stufe‡) n.

I have, in what follows, investigated certain properties of these functions, and functions derived from them, when n is a square number, and have treated in detail the particular cases where $n = 9$ and $n = 4$.

In the future, when there is no possible ambiguity, I shall write, instead of $X_{\frac{\alpha}{n}}(u \mid \omega_1, \omega_2)$, the briefer forms $X_\alpha(u \mid \omega_1, \omega_2)$ or $X_\alpha(u)$; and instead of

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* Hereafter I shall refer to this treatise as "Klein-Fricke."

† Klein-Fricke, Vol. II, p. 261.

‡ Klein, "Ueber die Normal Curven der n^{ter} Ordnung."

$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(u \mid \omega_1, \omega_2)$ the briefer forms $\sigma_{\lambda,\mu}(u)$ and $\sigma_{\lambda,\mu}(0) = \sigma_{\lambda,\mu}$. I shall enumerate without proof certain properties of the functions defined by (1), viz.

$$X_{\alpha+n}(u) = X_\alpha(u), \tag{3}*$$

$$X_\alpha(-u) = (-1)^n X_{n-\alpha}(u), \tag{4}*$$

$$X_\alpha\left(u + \frac{\lambda\omega_1 + \mu\omega_2}{n}\right) = (-1)^{n(\lambda+\mu)} e^{(\lambda\eta_1 + \mu\eta_2)\left(u + \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)} e^{\frac{\mu\pi i(\lambda - 2\alpha)}{n}} X_{\alpha-\lambda}(u), \tag{5}*$$

$$X_\alpha(u) = (-1)^{n\alpha} e^{\alpha\eta_1\left(u - \frac{\alpha\omega_1}{2n}\right)} X_0\left(u - \frac{\alpha\omega_1}{n}\right). \tag{6}*$$

There are n distinct functions $X_\alpha(u)$, and n only, and they are linearly independent.

There are two cases to be considered: Case I, $n \equiv 1 \pmod 2$; Case II, $n \equiv 0 \pmod 2$.

Case I.

1.—*Transformation of the $X(u)$ by the Modular Substitutions when n is Odd.*

The arbitrary constant C_α in $X_\alpha(u)$ may be so chosen that we may write, *in case n is odd,*

$$X_\alpha(u \mid \omega_1, \omega_2) = (-1)^\alpha \sqrt[8]{\frac{\Delta}{\Delta^n}}\, e^{-G_1 u^2} \sigma_{\frac{\alpha}{n}, 0}\left(u \mid \omega_1 \frac{\omega_2}{n}\right), \tag{7}†$$

or

$$X_\alpha(u \mid \omega_1, \omega_2) = (-1)^\alpha - i\frac{e^{\frac{u^2 n\eta_1}{2\omega_1}}}{\sqrt[8]{\Delta^n}} \cdot \sqrt{\frac{2\pi i}{\omega_1}}\, \vartheta_1\left(-\frac{\pi}{\omega_1}\ \frac{(u - \alpha\omega_1)}{n},\ e^{-2\frac{\pi i\omega_2}{n\omega_1}}\right), \tag{8}$$

‡ where $\Delta = \Delta\left(\omega_1, \frac{\omega_2}{n}\right)$ and

$$G_1 = \frac{\bar{\eta}_1 - n\eta_1}{2\omega_1} = \frac{\bar{\eta}_2 - n\eta_2}{2\omega_2},$$

where, owing to the cogredience‖ of ω_1, ω_2 and η_1, η_2,

$$\bar{\eta}_1 = \eta_1 \quad \text{and} \quad \bar{\eta}_2 = \frac{\eta_2}{n}.$$

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* Klein-Fricke, Vol. II, p. 264, et seq.

† Klein-Fricke, Vol. II, p. 277.

‡ For definition of Δ, see Klein-Fricke, Vol. I, p. 15.

‖ Klein-Fricke, Vol. I, p. 122.

Professor Klein has shown* that there exists a finite group of linear substitutions on $X_\alpha(u)$, of order $2\mu(n)$ which may be generated by repetitions and combinations of the operators S and T, which are defined as follows:

$$S:\ \omega_1' = \omega_1 + \omega_2,\quad \omega_2' = \omega_2, \qquad T:\ \omega_1' = -\omega_2,\quad \omega_2' = \omega_1, \tag{9}$$

and that if $\varepsilon = e^{\frac{2\pi i}{n}}$,

$$\left.\begin{aligned} S:&\ \ X_\alpha'(u) = \varepsilon^{\frac{-(n-\alpha)\alpha}{2}} X_\alpha(u), \\ T:&\ \ X_\alpha'(u) = \frac{i^{\frac{n-1}{2}}}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_\beta(u), \end{aligned}\right\} \tag{10}$$

Where S: $X_\alpha'(u)$ means that $X_\alpha(u)$ has been subjected to the operation S.

We shall now define a set of $(n-1)/2$ distinct functions† of ω_1 and ω_2 by the following equation:

$$X_\alpha(0) = z_\alpha(\omega_1, \omega_2) = z_\alpha \text{ (say)}. \tag{11}$$

If we put $u = 0$ and $n = 9$ in equations (10), we obtain a group of quaternary substitutions on the z_α generated by

$$\left.\begin{aligned} S:&\ \ z_\alpha' = \varepsilon^{\frac{-(9-\alpha)\alpha}{2}} z_\alpha, \\ T:&\ \ z_\alpha' = \tfrac{1}{3}\sum_{\beta=0}^{8} \varepsilon^{-\alpha\beta} z_\beta\,.\ (\alpha = 1\ \ldots .\ 4). \end{aligned}\right\} \tag{12}$$

The order of this group is $(9)\phi(9)\psi(9) = 648$. We shall call it $\overline{G}_{648}$.

2.—*The Biquadratic Relations of the* z_α.

If we substitute in the well-known σ-relation,‡

$$\begin{aligned} &\sigma(u_1+u_2)\,\sigma(u_3+u_4)\,\sigma(u_1-u_2)\,\sigma(u_3-u_4) \\ +&\sigma(u_1+u_3)\,\sigma(u_1-u_3)\,\sigma(u_4+u_2)\,\sigma(u_4-u_2) \\ +&\sigma(u_1+u_4)\,\sigma(u_1-u_4)\,\sigma(u_2+u_3)\,\sigma(u_2-u_3) = 0, \end{aligned}$$

* Klein-Fricke, Vol. II, p. 296.

† To show that there actually exists a set of functions of ω_1, ω_2 such as we have defined, see Klein-Fricke, Vol. II, p. 267, equation (12).

‡ Schwarz, Formeln und Lehrsätze zum Gebrauche der Elliptischen Functionen, p. 47.

$u_i = \frac{u}{2} - \frac{\alpha_i\omega_1}{n}$, we obtain, after reduction,

$$X_{\alpha_1+\alpha_2} X_{\alpha_3+\alpha_4} z_{\alpha_3-\alpha_1} z_{\alpha_4-\alpha_2} + X_{\alpha_1+\alpha_3} X_{\alpha_2+\alpha_4} z_{\alpha_2-\alpha_1} z_{\alpha_3-\alpha_4}$$
$$+ X_{\alpha_1+\alpha_4} X_{\alpha_2+\alpha_3} z_{\alpha_4-\alpha_1} z_{\alpha_3-\alpha_2} = 0. \quad \text{(a)*}$$

Put in this equation $u = 0$, and we obtain

$$z_{\alpha_1+\alpha_2} z_{\alpha_3+\alpha_4} z_{\alpha_3-\alpha_1} z_{\alpha_4-\alpha_2} + z_{\alpha_1+\alpha_3} z_{\alpha_4+\alpha_2} z_{\alpha_2-\alpha_1} z_{\alpha_3-\alpha_4}$$
$$+ z_{\alpha_1+\alpha_4} z_{\alpha_2+\alpha_3} z_{\alpha_4-\alpha_1} z_{\alpha_3-\alpha_2} = 0. \quad \text{(b)}$$

The left side of equation (b) vanishes for all values of ω_1, ω_2. It is our purpose to find all such expressions for the case $n = 9$. Inspection shows that the left side of equation (b) vanishes identically in z_α, if any subscript of z_α is congruent zero modulo n.

If we write $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = s$, then any pair of X's occurring in (a) may be put in the form $X_i . X_{s-i}$. Therefore, by virtue of equation (3), s runs through the residue system, modulo 9.

By virtue of equation (a), any X_α-pair $X_i X_{s-i}$ can be expressed in terms of two pairs, $X_k X_{s-k}, X_l X_{s-l}$, arbitrarily chosen; and any other relation can be obtained by the elimination of one of these pairs from two independent equations, provided the determinant of the coefficients of $X_k X_{s-k}$ and $X_l X_{s-l}$ does not vanish.

Choose then: the pairs $X_1 X_{s-1}$, $X_2 X_{s-2}$; whence

$$\left.\begin{aligned} \alpha_1 + \alpha_3 &\equiv 1, \\ \alpha_2 + \alpha_4 &\equiv s - 1, \\ \alpha_1 + \alpha_4 &\equiv 2, \\ \alpha_3 + \alpha_2 &\equiv s - 2, \end{aligned}\right\} \pmod{9}.$$

Making substitutions in equation (b) consistent with the foregoing, we obtain all relations except in the case $s = 3$, when the equation vanishes identically in z_α. They are

$$\begin{aligned} z_2^3 z_1 - z_3^3 z_2 + z_4^3 z_1 = 0 &= S_1 \text{ (say)}, \\ z_1^3 z_4 + z_2^3 z_4 - z_3^3 z_1 = 0 &= S_2 \text{ (say)}, \\ -z_1^2 z_2 + z_2^2 z_4 + z_1 z_4^2 = 0 &= S_3 \text{ (say)}. \end{aligned}$$

* Here, and throughout this section, X_α means $X_\alpha(u)$.

Since no relation exists in case $s=3$ involving the X_s-pair X_1X_2, there remains only one relation, and it is found to be

$$-z_2z_1^3+z_3^3z_4+z_2z_4^3=0=S_4 \text{ (say)}.$$

If we regard z_1, z_2, z_3, z_4 as homogeneous coordinates in a space of three dimensions, and if $S_1-z_4S_4=z_2S_5$, then the curve

$$S_4=0,\quad S_5=0$$

is the complete intersection of the four surfaces

$$S_1=0=S_2=S_3=S_4.$$

The curve is of ninth degree.

3.—*The Invariant Tetrahedron. The Functions t_a.*

Regarding again z_1, z_2, z_3, z_4 as homogeneous coordinates in a space of three dimensions, we shall show first, *that no plane is invariant under the substitutions of* $\bar{G}_{648}$. Let

$$A_1z_1+A_2z_2+A_3z_3+A_4z_4=0$$

be the equation of any plane. Then

$$*S:\quad A_1z_1+A_2z_2+A_3z_3+A_4z_4=\epsilon^5A_1z_1+\epsilon^2A_2z_2+A_3z_3+\epsilon^8A_4z_4.$$

Hence the only planes unchanged by S are $z_1=0$, $z_2=0$, $z_3=0$, $z_4=0$. None of these are unchanged by T. This proves the theorem.

Let us find now the equations of all tetrahedrons invariant under the group of substitutions $\bar{G}_{648}$.

Any tetrahedron invariant under S is evidently invariant under S^3. Let

$$A_1z_1+A_2z_2+A_3z_3+A_4z_4=0=P$$

be the equation of a plane of the invariant tetrahedron.

$$S^3:\quad A_1z_1'+A_2z_2'+A_3z_3'+A_4z_4'=A_1\epsilon^6z_1+A_2\epsilon^6z_2+A_3z_3+A_4\epsilon^6z_4=Q=0,$$
$$S^6:\quad A_1z_1'+A_2z_2'+A_3z_3'+A_4z_4'=A_1\epsilon^3z_1+A_2\epsilon^3z_2+A_3z_3+A_4\epsilon^3z_4=R=0.$$

* By this notation is meant that we apply the operation S to ΣA_iz_i and derive $\Sigma A_iz_i'$. But we know that this operation yields the expression on the right of the equation. We shall hereafter use this notation.

Since $$(S^3)^3 = 1,$$

the operation S^3 either leaves the planes $P=0$, $Q=0$ and $R=0$ each unchanged or permutes them. Hence S leaves at least one plane of the tetrahedron unchanged.

Suppose $$P_1 = A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = 0$$

be the equation of this plane, it is invariant under S if, and only if,

$$\left.\begin{array}{l} A_1 = A_2 = A_4 = 0 \\ \text{or} \quad A_3 = 0 . \end{array}\right\} \qquad \text{(A)}$$

Let us choose first, the first set of relations; then

$$z_3 = 0$$

is one of the planes of the tetrahedron. The equation of the other three planes may be found, if an invariant tetrahedron exists, by subjecting z_3 to any three operations of the group such as $T^\nu S^\mu$, $\nu \neq 0$. The equation of the tetrahedron is found to be

$$z_3(z_1 - z_2 + z_4)(\varepsilon^8 z_4 - z_2 + \varepsilon^6 z_4)(z_1 - \varepsilon^8 z_2 + \varepsilon^6 z_4) = 0. \qquad (13)$$

The plane $$A_1z_1 + A_2z_2 + A_4z_4 = 0 = L \text{ (say)},$$

is unchanged by S^3, and hence, reasoning as before, one plane at least of the tetrahedron is unchanged by S. The only planes unchanged by S are

$$z_1 = 0; \quad z_2 = 0, \quad z_4 = 0 \text{ and } z_3 = 0 .$$

The last one we have discussed. Let us examine the others: Let

$$\begin{aligned} T &: \ z_1 = L, \\ TS &: \ z_1 = M, \\ TS^2 &: \ z_1 = N. \end{aligned}$$

Since $S:N$ produces neither L, M nor z_1, the tetrahedron

$$z_1 LMN = 0$$

is not invariant under the substitutions of the group. Hence $z_1 = 0$ is not one of the planes of the tetrahedron. In a similar way, it may be shown that $z_2 = 0$ and $z_4 = 0$ are not planes of the tetrahedron. Hence, the only tetrahe-

dron invariant under $\overline{G}_{648}$ is given by equation (13). Put now

$$\left.\begin{aligned}3t_1 &= z_1 - z_2 + z_4,\\ 3t_2 &= \varepsilon^5 z_1 - \varepsilon^2 z_2 + \varepsilon^8 z_4,\\ 3t_3 &= \varepsilon z_1 - \varepsilon^4 z_2 + \varepsilon^7 z_4,\\ 3t_4 &= (\varepsilon^3 - \varepsilon^6) z_3.\end{aligned}\right\} \tag{14}$$

We have immediately

$$S:\ \left.\begin{aligned}t_1' &= t_2,\\ t_2' &= t_3,\\ t_3' &= \varepsilon^6 t_1,\\ t_4' &= t_4,\end{aligned}\right\}(15) \qquad T:\ \left.\begin{aligned}t_1' &= -\ t_4,\\ t_2' &= \varepsilon^3 t_3,\\ t_3' &= -\varepsilon^6 t_2,\\ t_4' &= t_1.\end{aligned}\right\}(16)$$

The substitutions defined by (15) and (16) form a group of unary substitutions, holoedric isomorphic with $\overline{G}_{648}$. I shall call this group G_{648}.

The functions $\sigma_{\frac{\lambda}{3},\frac{\mu}{3}}$ when subjected to S and T defined by (9) are interchanged among themselves as follows:

$$S:\ \begin{aligned}\sigma_{0,1}' &= \sigma_{0,1},\\ \sigma_{1,0}' &= \sigma_{1,1},\\ \sigma_{1,1}' &= \sigma_{1,2},\\ \sigma_{1,2}' &= \varepsilon^6\sigma_{1,0},\end{aligned} \qquad T:\ \begin{aligned}\sigma_{0,1}' &= \sigma_{1,0},\\ \sigma_{1,0}' &= -\ \sigma_{0,1},\\ \sigma_{1,1}' &= \varepsilon^3\sigma_{1,2},\\ \sigma_{1,2}' &= -\varepsilon^6\sigma_{1,1}.\end{aligned}$$

Comparing the substitutions on the $\sigma_{\frac{\lambda}{3},\frac{\mu}{3}}$ with those* on $t_{\frac{a}{9}}$, we see that $\sigma_{\frac{1}{3},\frac{2}{3}}$, $\sigma_{\frac{1}{3},\frac{1}{3}}$, $\sigma_{\frac{1}{3},\frac{2}{3}}$, $\sigma_{\frac{2}{3},\frac{1}{3}}$, are cogredient† with $t_{\frac{1}{9}}, t_{\frac{2}{9}}, t_{\frac{3}{9}}, t_{\frac{4}{9}}$.

The functions $\sigma_{\frac{\lambda}{3},\frac{\mu}{3}}$ and $t_{\frac{a}{9}}$ are functions of ω_1, ω_2.

We may find the relations connecting them by considering the theory of the transformation of the θ-functions in terms of which both may be expressed. They are:

$$\left.\begin{aligned}\sigma_{\frac{2}{3},\frac{1}{3}} &= \Delta t_{\frac{4}{9}},\\ \sigma_{\frac{1}{3},\frac{2}{3}} &= \Delta t_{\frac{1}{9}},\\ \sigma_{\frac{1}{3},\frac{1}{3}} &= \Delta t_{\frac{2}{9}},\\ \sigma_{\frac{1}{3},\frac{2}{3}} &= \Delta t_{\frac{3}{9}}.\end{aligned}\right\} \tag{α}$$

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* Since t_a are derived from z_a and z_a from $X_a(u)$, when it is especially desirable to emphasize the relation of a to 9, we shall write $t_{\frac{a}{9}}$ instead of t_a and $z_{\frac{a}{9}}$ instead of z_a.

† Burnside and Panton, Theory of Equations, p. 384.

4.—*The Invariant Functions of t_α.*

By an invariant function we shall mean a rational integral function of t_α, defined by equation (14), which does not vanish identically in t_α, with degree greater than zero, and which remains unchanged by all the substitutions of G_{648}. Such functions exist, for the elementary symmetric functions of t_1^6, t_2^6, t_3^6, t_4^6 are invariant under this group.

An i-lettered term is one containing i letters.

An i-lettered function is a rational integral function containing only i-lettered terms.

A form is a homogeneous integral function. We shall now seek all invariant functions of t_α.

Any invariant integral function F may be written $F = \sum_{\kappa=0}^{4} a_\kappa F_\kappa$, where F_κ are i-lettered functions and a_κ are independent of t_α and do not all vanish. Since, under any substitution of G_{648}, we change neither the number of letters nor the degree of any term, it follows that each F_κ consists of the sum of i-lettered forms, and is, therefore, itself an invariant form. Any four-lettered form is a rational integral function of $t_1 t_2 t_3 t_4$ and one-, two- and three-lettered forms.

We shall now derive all three-lettered invariant forms. Choose arbitrarily one of the terms R_i, where

$$R_1 = t_1^\alpha t_2^\beta t_3^\gamma, \quad R_2 = t_1^\alpha t_2^\beta t_4^\gamma, \quad R_3 = t_1^\alpha t_3^\beta t_4^\gamma, \quad R_4 = t_2^\alpha t_3^\beta t_4^\gamma, \tag{a}$$

where α, β, γ are integers. Call the term so chosen the originating term and denote it by R. Let the terms derived by subjecting R to the substitutions of G_{648} be denoted by R_i. It is evident that R_1, R_2, R_3 and R_4 will be of their number. Let $F_{\alpha,\beta,\gamma}$ be an invariant three-lettered function of degree $\alpha + \beta + \gamma$. Then $F_{\alpha,\beta,\gamma}$ *is the simplest symmetric function* of R_i; for, since it is unchanged by S and T, it is unchanged by $S^\mu T^\nu$. Therefore, the R_i are all found among the terms of $F_{\alpha,\beta,\gamma}$. Moreover, $F_{\alpha,\beta,\gamma}$ is of the form

$$F_{\alpha,\beta,\gamma} = \Sigma a_i R_i,$$

where a_i is independent of t_α. For, $F_{\alpha,\beta,\gamma}$ is a form, and any operation except the addition of terms such as $a_i R_i$ either destroys its homogeneity or changes its

degree. Some substitution σ of G_{648} exists that changes R_i into R_j, and the necessary and sufficient condition that $\sigma : F'_{\alpha,\beta,\gamma} = F_{\alpha,\beta,\gamma}$ is that $a_i = a_j$.

Now, if R_i be either of the terms (a), then

$$T^2 : R_i = (-1)^{(\alpha+\beta+\gamma)} R_i.$$

Hence, when we form the simplest symmetric function of the originating term and those obtained by operating with T^2, this function vanishes identically unless

$$\alpha + \beta + \gamma \equiv 0 \pmod 2. \tag{b}$$

Hence, we conclude that this congruence holds.

Also, if

$$\begin{aligned} & R = t_1^\alpha t_2^\beta t_3^\gamma, \\ S^3: \quad & R' = \epsilon^{6(\alpha+\beta+\gamma)} R, \\ S^6: \quad & R' = \epsilon^{3(\alpha+\beta+\gamma)} R. \end{aligned}$$

Hence, the simplest symmetric function formed from the originating term and the terms resulting from the repeated applications of S^3, vanishes identically unless

$$\alpha + \beta + \gamma \equiv 0 \pmod 3. \tag{c}$$

Hence, we conclude that congruence (c) holds; or that no three-lettered invariant function exists. But we have shown that some do exist.

If we choose R_2 as the originating term, we may show in a similar manner that

$$\alpha + \beta \equiv 0 \pmod 3,$$

therefore,

$$\gamma \equiv 0 \pmod 3.$$

In a similar way we can show that

$$\alpha \equiv 0 \pmod 3$$

and

$$\beta \equiv 0 \pmod 3.$$

We conclude, therefore, that in all three-lettered forms the exponents satisfy the congruences

$$\left.\begin{aligned} \alpha + \beta + \gamma \equiv 0 \pmod 6), \\ \alpha \equiv \beta \equiv \gamma \equiv 0 \pmod 3). \end{aligned}\right\} \tag{d}$$

Choosing as our originating term $t_1^\alpha t_2^\beta t_3^\gamma$, we obtain as a three-lettered inva-

riant form

$$\left.\begin{aligned} F_{\alpha,\beta,\gamma} = {} & t_1^\alpha t_2^\beta t_3^\gamma + t_2^\alpha t_3^\beta t_1^\gamma + t_3^\alpha t_1^\beta t_2^\gamma \\ & + (-1)^\alpha [t_1^\alpha t_3^\beta t_4^\gamma + t_2^\alpha t_1^\beta t_4^\gamma + t_3^\alpha t_2^\beta t_4^\gamma] \\ & + (-1)^\beta [t_4^\alpha t_1^\beta t_3^\gamma + t_4^\alpha t_2^\beta t_1^\gamma + t_4^\alpha t_3^\beta t_2^\gamma] \\ & + (-1)^\gamma [t_3^\alpha t_4^\beta t_1^\gamma + t_1^\alpha t_4^\beta t_2^\gamma + t_2^\alpha t_4^\beta t_3^\gamma]. \end{aligned}\right\} \tag{17}$$

Giving to α, β, γ in (17) all values satisfying the congruences (d), we obtain all possible three-lettered invariant forms. For, by making suitable interchanges of α, β, γ, we can find among the terms of (17) any of the four possible originating terms.

If we put in (17) one exponent, $\gamma = 0$, we get all two-lettered invariant functions,

$$\left.\begin{aligned} F_{\alpha,\beta} = {} & t_1^\alpha t_2^\beta + t_2^\alpha t_3^\beta + t_3^\alpha t_1^\beta \\ & + (-1)^\alpha (t_1^\alpha t_3^\beta + t_2^\alpha t_1^\beta + t_3^\alpha t_2^\beta) \\ & + (-1)^\beta (t_4^\alpha t_1^\beta + t_4^\alpha t_2^\beta + t_4^\alpha t_3^\beta) \\ & + (t_3^\alpha t_4^\beta + t_1^\alpha t_4^\beta + t_2^\alpha t_4^\beta). \end{aligned}\right\} \tag{18}$$

Put in (18), $\beta = 0$ and $F_{\alpha,0} = F_\alpha$, we obtain

$$F_\alpha = 3(t_1^\alpha + t_2^\alpha + t_3^\alpha + t_4^\alpha). \tag{19}$$

5.—*The Basis of the System of Forms.*

Giving to α, β, γ all possible values satisfying the congruences (d), we obtain from equations (17)–(19) an infinity of forms, all invariant under the group G_{648}. And, as we have shown that they are the only forms so constituted, we have a totality of forms definitely defined. We shall designate this totality of forms by H. By Hilbert's *Law we can express any form, F, of H, in the following form:

$$F = \sum_{i=1}^{\mu} A_i F_i, \text{ where } \mu \text{ is finite,}$$

†and A_i are forms of H. The forms F_i is called the basis of the system H.

* Hilbert, "Ueber die Theorie der Algebraischen Formen," in Mathematische Annalen, Band 36. Weber, Lehrbuch der Algebra, Band 2, p. 165.

† Weber, Lehrbuch der Algebra, Band 2, p. 170.

Problem : *To find the basis of H.*

In order to solve the problem, we need the set of identities (a) .... (g), the truth of which may be verified by computation. The summation under the summation signs, and the products under the product signs, are made from 1 .... 4, although it is not so indicated in the equations of the present section.

$$\Sigma t_i^6 . F_{\alpha,\beta,\gamma} = F_{\alpha,\beta+6,\gamma} + F_{\alpha,\beta,\gamma+6} + F_{\alpha+6,\beta,\gamma} + \Pi t_i^6 . F_{\alpha-6,\beta-6,\gamma-6}, \qquad \text{(a)}$$

$$\Sigma t_i^6 t_k^6 . F_{\alpha,\beta,\gamma} = F_{\alpha,\beta+6,\gamma+6} + F_{\alpha+6,\beta,\gamma+6} + F_{\alpha+6,\beta+6,\gamma} + \Pi t_i^6 [F_{\alpha,\beta-6,\gamma-6} + F_{\alpha-6,\beta,\gamma-6} + F_{\alpha-6,\beta-6,\gamma}], \qquad \text{(b)}$$

$$\Sigma t_i^6 t_k^6 t_l^6 . F_{\alpha,\beta,\gamma} = F_{\alpha+6,\beta+6,\gamma+6} + \Pi t_i^6 [F_{\alpha-6,\beta,\gamma} + F_{\alpha,\beta-6,\gamma} + F_{\alpha,\beta,\gamma-6}, \qquad \text{(c)}$$

Therefore,

$$\Sigma t_i^6 . F_{\alpha,\beta,\gamma} - \Sigma t_i^6 t_k^6 . F_{\alpha-6,\beta,\gamma} + \Sigma t_i^6 t_k^6 t_l^6 . F_{\alpha-12,\beta,\gamma} - \Pi t_i^6 . F_{\alpha-18,\beta,\gamma} = F_{\alpha+6,\beta,\gamma}, \qquad \text{(d)}$$

$$\Sigma t_i^6 . F_{\alpha,\beta,\gamma} - \Sigma t_i^6 t_k^6 . F_{\alpha,\beta-6,\gamma} + \Sigma t_i^6 t_k^6 t_l^6 . F_{\alpha,\beta-12,\gamma} - \Pi t_i^6 . F_{\alpha,\beta-18,\gamma} = F_{\alpha,\beta+6,\gamma}, \qquad \text{(e)}$$

$$\Sigma t_i^6 . F_{\alpha,\beta,\gamma} - \Sigma t_i^6 t_k^6 . F_{\alpha,\beta,\gamma-6} + \Sigma t_i^6 t_k^6 t_l^6 . F_{\alpha,\beta,\gamma-12} - \Pi t_i^6 . F_{\alpha,\beta,\gamma-18} = F_{\alpha,\beta,\gamma+6}. \qquad \text{(f)}$$

$$F_{\alpha,\beta,\gamma} = F_{\beta,\gamma,\alpha} = F_{\gamma,\alpha,\beta}. \qquad \text{(g)}$$

Making $\alpha = 0$ in the equations (a), (b), (c), (d), (e), (f) and (g), we derive similar equations for $F_{\beta,\gamma}$.

Assuming $\alpha \equiv 0 \pmod 6$, the discussion is divided into two cases, according as

$$\beta \equiv \gamma \equiv 0 \pmod 2 \quad \text{or} \quad \beta \equiv \gamma \equiv 1 \pmod 2.$$

Case I.

$F_{\alpha,\beta}$ and $F_{\alpha,\beta,\gamma}$ are symmetric with regard to t_i^6; and each is accordingly a rational integral function* of the elementary symmetric functions of t_i^6.

Case II.

We shall consider

a. Two-lettered forms.

Making $\alpha = 0$ in equation (e), we can express every $F_{\beta,\gamma}$ as an integral function of $F_{\beta',\gamma}$ and $F_{\beta+3,3,3}$, where $\beta' \leqq 15$. Since $F_{\beta,\gamma} = -F_{\gamma,\beta}$, we have

* Chrystal's Algebra, Vol. I, p. 440.

an expression for every $F_{\beta,\gamma}$ in terms of $F_{\beta',\gamma'}$ and $F_{\beta+3,3,3}, F_{\gamma+3,3,3}$, where $\beta', \gamma' \leqq 15$. Whence β', γ' can have the following sets of values:

$$3,\ 9;\quad 3,\ 15;\quad 9,\ 15.$$

But by (b), $F_{9,15}$ can be expressed as an integral function of $F_{3,9}$ and certain $F_{\alpha,\beta,\gamma}$, which shall be discussed under b.

b. Three-lettered forms.

By successive applications of the recursion formulæ (d), (e), (f) and formula (a), we can express any $F_{\alpha,\beta,\gamma}$ as an integral function of $F_{9,3}$, $F_{15,3}$, the elementary symmetric functions of t_i^6 and $F_{\alpha',\beta',\gamma'}$, where the simultaneous values of α', β', γ' are given by

$$\begin{array}{llllll} \alpha' = 6; & 6; & 6; & 6; & 6; & 6; \\ \beta' = 3; & 3; & 3; & 15; & 15; & 15; \\ \gamma' = 3; & 9; & 15; & 3; & 9; & 15. \end{array}$$

We shall now show that every $F_{\alpha',\beta',\gamma'}$ can be expressed as an integral function of the elementary symmetric function of t_i^6, $F_{9,3}$, $F_{15,3}$, $F_{6,3,3}$, and $F_{6,9,3}$, and that none of the functions just named can be expressed as integral functions of forms of H of lower degree.

To express rationally and integrally one of these forms in terms of forms of lower degree, it must be involved in an equation with them; its coefficient must be independent of t_i, and, since the equation must be homogeneous, the coefficients of the forms of lower degree are functions of t_i. The method is sufficiently well illustrated by solving one case—that in which the degree of the form is 30.

We have the following set of equations:

$$\begin{aligned} &(\Sigma t_i^6)\, F_{6,9,9} = F_{6,15,9} + F_{12,9,9} + F_{6,9,15}, \\ &(\Sigma t_i^6 t_k^6)\, F_{6,3,3} - \Pi t_i^6 (F_{6,9,3} - F_{6,3,9}) = F_{12,9,9}, \\ &(\Sigma t_i^6 t_k^6 t_e^6)\, F_{9,3} - \Pi t_i^6 F_{12,3,3} = F_{6,15,9} - F_{6,9,15}, \end{aligned}$$

which enable us to solve for all forms of degree 30.

Accordingly the *elementary symmetric functions of* t_i^6, $F_{9,3}$, $F_{15,3}$, $F_{6,3,3}$, and $F_{6,9,3}$ *constitute the basis of the System* H.

6.—*The Invariant Forms as Rational Functions of g_2, g_3 and Δ.*

Each invariant function $F_{\alpha,\beta,\gamma}$ is a rational* function of J, where $J=\frac{g_2^3}{\Delta}$, and, therefore, a rational function of g_3 and Δ, where g_3 has the usual signification.†

By equations (α, p. 12), and the relations found in Klein-Fricke, Vol. II, p. 30, we obtain

$$\Delta^3 t_1^3 = \sigma_{1,2}^3 = \frac{\xi_3-\xi_4}{-i\sqrt[3]{\Delta}},$$

$$\Delta^3 t_2^3 = \sigma_{1,4}^3 = \frac{\rho^2\xi_3-\xi_4}{-i\sqrt[3]{\Delta}},$$

$$\Delta^3 t_3^3 = \sigma_{1,3}^3 = \frac{\rho\xi_3-\xi_4}{-i\sqrt[3]{\Delta}},$$

$$\Delta^3 t_4^3 = \sigma_{3,4}^3 = -\frac{\xi_4\sqrt{3}}{\sqrt[3]{\Delta}},$$

where $\rho = e^{\frac{2\pi i}{3}}$.

‡Whence

$$\left.\begin{aligned}
\Sigma t_i^6 &= 0,\\
\Sigma t_i^6 t_k^6 &= \frac{18}{\Delta^{18}},\\
\Sigma t_i^6 t_k^6 t_e^6 &= -\frac{216 g_3}{\Delta^{20}},\\
t_1^3 t_2^3 t_3^3 t_4^3 &= \frac{3\sqrt{3}}{i\Delta^{18}}
\end{aligned}\right\} \qquad (20)$$

and

$$F_{9,3} = -\frac{36}{i\Delta^{18}},$$

$$F_{15,3} = -\frac{3\cdot 216\sqrt{3}g_3}{\Delta^{20}},$$

$$F_{6,3,3} = -\frac{36}{i\Delta^{18}},$$

$$F_{6,9,3} = -\frac{3\rho 216 g_3}{\Delta^{20}}.$$

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* Weber, Elliptische Functionen, p. 149.

† Klein-Fricke, Vol. I, p. 15.

‡ Klein-Fricke, Vol. I, p. 680.

From equations (20), it is evident that t_1^6, t_2^6, t_3^6, t_4^6 are the roots of the equation

$$y^4 + \frac{18}{\Delta^{13}} y^2 + \frac{216 g_3}{\Delta^{20}} y - \frac{27}{\Delta^{26}} = 0.$$

Case II. $n \equiv 0 \pmod 2$.

7.—*The Functions* $X_a(u \mid \omega_1, \omega_2)$. *The Functions* z_a *for Case* $n \equiv 0 \pmod 2$.

The arbitrary constant C_a, in the definition of $X_a(u)$, may be determined so that we may write,* in case n is even,

$$X_a(u \mid \omega_1, \omega_2) = e^{-\frac{\pi i a}{n} + \frac{3\pi i}{4}} \sqrt{\Delta \,.\, \Delta} e^{-G_1 u^2} \sigma_{\frac{a}{n}+\frac{1}{2}, \frac{1}{2}} \left(u \mid \omega_1, \frac{\omega_2}{n}\right)$$

or,

$$X_a(u \mid \omega_1, \omega_2) = \sqrt{\frac{2n\pi}{\omega_2}} \sqrt[8]{\Delta} e^{\frac{n\eta_2 u^2}{2\omega_2}} \,.\, z^{-a} r^{\frac{a}{2n}} \theta_3 \left(\frac{nu - a\omega_1}{\omega_2} \pi, r^n\right).$$

There are in this case also n linearly independent functions $X_a(u)$ and no more.

Professor Klein has shown† that the linear substitutions on $X_a(u)$ corresponding to the modular substitutions S and T (defined in equation (9)), to be

$$\left.\begin{aligned} S:\quad & X_a'(u) = \varepsilon^{\frac{n}{8} + \frac{a^2}{2}} X_a(u \mid \omega_1, \omega_2), \\ T:\quad & X_a'(u) = \frac{-1}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-a\beta} X_\beta(u \mid \omega_1, \omega_2). \end{aligned}\right\} \tag{21}$$

If we put

$$z_a = X_a(0),$$

there are in the case $n = 4$ three distinct z_a, viz. z_0, z_1, z_2. Putting $n = 4$ and $u = 0$ in equation (21) and writing $\varepsilon = e^{\frac{2\pi i}{8}}$, we obtain

$$\left.\begin{aligned} S:\quad & z_0' = \varepsilon z_0, \\ & z_1' = \varepsilon^3 z_1, \\ & z_2' = \varepsilon^5 z_2, \\ T:\quad & z_1' = -\tfrac{1}{2}(z_0 + 2z_1 + z_2), \\ & z_2' = -\tfrac{1}{2}(z_0 - z_2), \\ & z_3' = -\tfrac{1}{2}(z_0 - 2z_1 + z_2). \end{aligned}\right\} \tag{22}$$

* Klein-Fricke, Vol. II, p. 285. † Ib.

We shall designate these substitutions on z_a as S and T. By composition of S and T, we obtain a group of linear substitutions on three variables of order $4\phi(8)\psi(8) = 192$. We shall call this group $\overline{G}_{192}$.

If we regard z_0, z_1, z_2 as homogeneous coordinates in a plane, we may show exactly as in Case I, that *no straight line is unchanged under the substitutions S and T*, and that

$$z_1(z_0 - z_2)(z_0 + z_2) = 0$$

is a triangle invariant under $\overline{G}_{192}$, and the only one.

Put

$$\begin{aligned} u_0 &= z_0 + z_2, \\ u_1 &= z_0 - z_2, \\ u_2 &= 2z_1. \end{aligned}$$

Then, from equations (22), we derive the substitution group of the u_a. It is given by

$$\left.\begin{aligned} S:\quad u_0' &= \varepsilon u_1, \\ u_1' &= \varepsilon u_0, \\ u_2' &= \varepsilon^2 u_2, \\ T:\quad u_0' &= -u_0, \\ u_1' &= -u_2, \\ u_2' &= -u_1. \end{aligned}\right\} \tag{23}$$

We shall call the group generated by equations (23) G_{192}.

By a consideration of the theory of transformation of the θ-functions, we may find the relation between $u_{\frac{a}{4}}$ and $\sigma_{\frac{\lambda}{2},\frac{\mu}{2}}$. They are

$$\left.\begin{aligned} \sigma_{\frac12,\frac12} &= \frac{(1+i)}{2\sqrt{2}}\,\frac{(u_{\frac{0}{4}})}{\sqrt[4]{\Delta}}, \\ \sigma_{\frac12,\frac02} &= \frac{(1+i)}{2\sqrt{2}}\,\frac{\varepsilon u_{\frac14}}{\sqrt[4]{\Delta}}, \\ \sigma_{\frac02,\frac12} &= \frac{(1+i)}{i2\sqrt{2}}\,\frac{\varepsilon u_2}{\sqrt[4]{\Delta}}. \end{aligned}\right\} \tag{24}$$

8.—*The Invariant Functions of u_a.*

We shall now seek the functions of u_a which are invariant under the substitutions of G_{192}.

Reasoning, as in Case I, we can show that every invariant form is an integral function of two-lettered invariant forms, one-lettered invariant forms, and $(u_0u_1u_2)^\alpha$, where α is a positive integer. Choosing as our originating term $u_0^\alpha\, u_1^\beta$ and forming the simplest symmetric function of terms arising from subjecting it to the operators of G_{192}, we may show by reasoning, as in Case I, that every two-lettered invariant form may be obtained by giving to α, β all values in $F_{\alpha\beta}$, where

$$\alpha \equiv 0 \pmod 4),$$
$$\beta \equiv 0 \pmod 4),$$

and where

$$F_{\alpha\beta} = u_0^\alpha u_1^\beta + \varepsilon^{\alpha+\beta} u_1^\alpha u_0^\beta + u_0^\alpha u_2^\beta + \varepsilon^{\alpha+\beta} u_2^\alpha u_0^\beta + \varepsilon^\alpha u_1^\alpha u_2^\beta + \varepsilon^\alpha u_2^\alpha u_1^\beta. \tag{25}$$

Making $\beta = 0$ in (25), we get all invariant one-lettered functions, and if we write $F_{\alpha,0} = F_\alpha$ we get

$$F_\alpha = 2\,(u_0^\alpha + \varepsilon^\alpha u_1^\alpha + \varepsilon^\alpha u_2^\alpha). \tag{26}$$

We shall now seek the basis of this system of forms obtained by giving α, β all values in (25) and (26).

By calculation it is shown that

$$F_{4,4}\,F_{\alpha,\beta} - F_{\alpha,\beta+4}\,F_4 = F_{\alpha,\beta+8} + (u_0\,u_1\,u_2)^4\,(F_{\alpha,\beta-4}). \tag{a}$$

This is a recursion formula for β, and since

$$F_{\alpha,\beta} = \varepsilon^{\alpha+\beta}\,F_{\beta,\alpha},$$

it is also a recursion formula for α.

By the use of (a), we can express every $F_{\alpha,\beta}$ as an integral function of $F_{\alpha',\beta'}$, where $\alpha', \beta' \leq 8$. We have left, then, the form $F_{\alpha',\beta'}$, where α', β' may have the pairs of values 0, 4; 0, 8; 4, 4; 4, 8; 8, 8. But

$$2F_4^2 = F_8 - 2F_{4,4},$$
$$F_4\,F_{4,4} = 4F_{8,4} - 2\Pi u_i^4,$$
$$F_{4,4}^2 = 2F_{8,8} + \Pi u_i^4 . F_4.$$

The basis of the system is, therefore, $F_4, F_{4,4}$ and $(u_0\,u_1\,u_2)^2$.

Since these forms are invariant under S and T, each of these (and, therefore, every invariant function) can be expressed as a rational function of g_2, g_3 and Δ. We shall now find F_4, $F_{4,4}$, $(u_0\,u_1\,u_2)^2$ in terms of these invariants.

* Since $\sigma^4_{1,1} + \sigma^4_{0,1} + \sigma^4_{1,0} = 0$,

$$F_4 = 0.$$

† Now,

$$\sigma^4_{0,1} = -8\lambda_4,$$
$$\sigma^4_{1,0} = 8\lambda_3,$$
$$\sigma^4_{1,1} = -8(\lambda_3 - \lambda_4).$$

Therefore, from equations (24),

$$u_0^4 = 2^4\Delta 8(\lambda_3 - \lambda_4),$$
$$u_1^4 = 2^4\Delta 8\lambda_3.$$
$$u_2^4 = -2^4\Delta 8\lambda_4.$$

‡ Therefore,

$$F_{4.4} = 2^{14}\Delta g_2,$$
$$u_0^4 u_1^4 u_2^4 = 2^{22}\Delta^2.$$

Hence u_0^4, u_1^4, u_2^4 satisfies the equation

$$x^3 + 2^{14}\Delta g_2 x - 2^{32}\Delta^3 = 0.$$

Now, $u_0^2 u_1^2 u_2^2 = \left(\frac{2\pi}{\omega_2}\right)^3 \frac{\Delta}{\Delta^{\frac{1}{4}}} \cdot r^{\frac{1}{4}}$, all multiplied into a power series of integral powers of r with real coefficients $= \left(\frac{2\pi}{\omega_2}\right)^3 \Delta$ into a power series of integral powers of r with real coefficients.

If we now approach infinity along positive y-axis, we see that $u_0^2 u_1^2 u_2^2$ has the same sign as Δ. Therefore, we may write

$$u_0^2 u_1^2 u_2^2 = 2^{11}\Delta.$$

9.—*Linear Connection between $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ and $X_{\frac{a}{n^2}}(u)$.*

We shall show now that $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ is a linear homogeneous function of $X_{\frac{a}{n^2}}(u)$ for any finite value of n.

$$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu) = F(\omega, \eta)\, e^{\left[\left(\lambda + \frac{n(n-1)}{2}\right)\eta_1 + \left(\mu + \frac{n(n-1)}{2}\right)\eta_2\right]\left(u - \frac{(\lambda + (n-1)n)\omega_1 + (\mu + (n-1)n)\omega_2}{2}\right)}$$

$$\cdot \prod_{m_1, m_2} \sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{n^2} - \frac{m_1\omega_1 + m_2\omega_2}{n}\right), \qquad (m_1, m_2 = 0 \ldots\ldots n-1).$$

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* Klein-Fricke, Vol. II, p. 29. † Ib., Vol. I, p. 628. ‡ Klein-Fricke, Vol. I, p. 629.

This is a σ-product of n^2 factors multiplied by e with an exponent in which the coefficient of $u = \left[\lambda + \frac{n(n-1)}{2}\right]\eta_1 + \left[\mu + \frac{n(n-1)}{2}\right]\eta_2$, and whose residual sum $s = \lambda\omega_1 + \mu\omega_2 + \frac{n(n-1)}{2}\omega_1 + \frac{n(n-1)}{2}\omega_2$; moreover, there are n^2 different quantities $\sigma_{\frac{\lambda}{n},\frac{\mu}{n}}(nu)$.

If we define n^2 functions by the equations

$$x_i = c_i \prod_{k=0}^{n^2-1} \sigma(u - a_{i,k}). \qquad (i = 0 \ldots . n^2 - 1),$$

where c_i is independent of u and the residual sum,

$$s = \sum_{k=0}^{n^2-1} a_{i,k} = 0, \tag{a}$$

then by *Hermites' Law $\sigma_{\frac{\lambda}{n},\frac{\mu}{n}}(nu)$ can be expressed as a linear homogeneous function of x_i.

Also, if n is odd,

$$X_{\frac{a}{n^2}}(u) = f(\omega, \eta)\, e^{\left(a\eta + \left(\frac{n^2-1}{2}\right)\eta_2\right)\left(u - a\omega_1 + \frac{n^2-1}{2}\omega_2\right)} \cdot \prod_{\mu=0}^{n^2-1} \sigma\left(u - \frac{a\omega_1 + \mu\omega_2}{n^2}\right).$$

From this it appears that $X_{\frac{a}{n^2}}(u)$ is a σ-product of n^2 factors multiplied by e with an exponent which has the coefficient of $u = a\eta_1 + \frac{(n^2-1)}{2}\eta_2$ and whose residual sum

$$s = a\omega_1 + \frac{n^2-1}{2}\omega_2.$$

There are n^2 different quantities $X_{\frac{a}{n^2}}(u)$, and hence, according to Hermite's Law, $X_{\frac{a}{n^2}}(u)$ can be expressed as a linear homogeneous function of x_i defined by equa-

* Klein, Elliptischen Normal Curven der n^{ter} Ordnung, p. 854.

tion (a), and hence $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ can in general be expressed as a linear homogeneous function of $X_{\frac{\alpha}{n^2}}(u)$.

If n is even,

$$X_{\frac{\alpha}{n^2}}(u) = \phi(\omega, \eta)\, e^{\left(n^2\left(\frac{\alpha}{n^2}+\frac{1}{2}\right)\eta_1 + \frac{n^2}{2}\eta_2\right)\left(u - \frac{n^2\left(\frac{\alpha}{n^2}+\frac{1}{2}\right)\omega_1 + \frac{n^2}{2}\omega_2}{2}\right)}$$

$$\cdot \prod_{\mu} \sigma\left(u - \left[\left(\frac{\alpha}{n^2} + \tfrac{1}{2}\right)\omega_1 + \frac{\mu + \frac{1}{2}}{n^2}\omega_2\right]\right).$$

Reproducing the argument used above, it follows that, in this case also, $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ can be expressed as a linear homogeneous function of $X_{\frac{\alpha}{n^2}}(u)$, and hence for any finite value of n we may write

$$\sigma_{\frac{0}{n}, \frac{1}{n}}(nu) = \sum_{\alpha=1}^{n^2-1} A_\alpha X_{\frac{\alpha}{n^2}}(u), \tag{27}$$

where A_α is independent of u.

We shall now determine A_α of equation (27) in the case n is even. The theory of the transformation of the theta-functions gives

$$\sigma_{\frac{\frac{n}{2}}{n}, \frac{\frac{n}{2}-\alpha}{n}}(nu) = \frac{-\frac{1+i}{\sqrt{2}}}{n\sqrt[4]{\Delta}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_{\frac{n\beta}{n^2}}(u). \tag{28}$$

By equation (7) (Klein-Fricke, Vol. II, p. 27), the $\sigma_{\frac{\frac{n}{2}}{n}, \frac{\frac{n}{2}-\alpha}{n}}(nu)$ is transformed into another $\sigma_{\frac{\lambda'}{n}, \frac{\mu'}{n}}(nu)$ by both S^k and T^l. We shall indicate this transformation by marking the transformed quantities with a prime.

If we operate on the left side of equation (28) with T, we obtain

$$T\colon \quad X'_{\frac{n\beta}{n^2}}(u) = \sum_{\gamma=0}^{n^2-1} \varepsilon'^{-n\beta\gamma} X_{\frac{\gamma}{n^2}}(u),$$

where

$$\varepsilon' = e^{\frac{2\pi i}{n^2}}.$$

Whence,

$$T: \quad \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X'_{\frac{n\beta}{n^2}} = \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} \sum_{\gamma=0}^{n^2-1} \varepsilon^{j-n\beta\gamma} X_{\frac{\gamma}{n^2}}(u)$$

$$= X_{\frac{0}{n^2}}(u)\Big(\sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta}\Big) + \dots\dots X_{\frac{r}{n^2}}(u)\Big(\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(r+\alpha)}\Big)$$

$$+ \dots\dots X_{\frac{n^2-1}{n^2}}(u)\Big(\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(n^2-1+\alpha)}\Big). \qquad (29)$$

It is evident that

$$\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(r+\alpha)} = 0$$

if* $[(r+\alpha), n] = 1$; for, if f is the least number such that

$$\varepsilon^{-(r+\alpha)f} = 1 = \varepsilon^{-sn},$$

then, $f = \dfrac{sn}{r+\alpha}$. Hence, since $f \leq n$, $s = r + \alpha$. Therefore, $f = n$.

Hence $\varepsilon^{-(r+\alpha)}$ appertains† to the exponent n, and hence $\varepsilon^{-a(r+\alpha)}$ is an n^{th} root of unity, and if $a \neq a'$, $\varepsilon^{-a(r+\alpha)} \neq \varepsilon^{-a'(r+\alpha)}$, where $(a, a' = 0 \dots\dots n-1)$. Moreover, if

$$[(r+\alpha), n] = d, \quad d < n,$$

$\varepsilon^{-(r+\alpha)}$ is an n^{th} root of unity which appertains to the exponent $\dfrac{n}{d}$. Hence there are $\dfrac{n}{d}$ different quantities $\varepsilon^{-k(r+\alpha)}$, $\Big(k = 0 \dots\dots \Big(\dfrac{n}{d} - 1\Big)\Big)$. Therefore,

$$\sum_{k=0}^{\frac{n}{d}-1} \varepsilon^{-k(r+\alpha)} = 0. \qquad \text{(a)}$$

Since

$$\varepsilon^{-k(r+\alpha)} = \varepsilon^{-l(r+\alpha)},$$

if, and only if, $k \equiv l \Big(\text{mod } \dfrac{n}{d}\Big)$, and since there are just d quantities $< n$ satisfying this congruence, the quantities $\varepsilon^{-\beta(r+\alpha)}$ may be arranged in d sets, each of

* This notation has the ordinary signification that the G. C. D. of $r + \alpha$, and n is 1.

† Mathews, Theory of Numbers, Part I, p. 185.

which is defined by the summation (a). Therefore,

$$\Sigma \epsilon^{-\beta(r+\alpha)} = 0 \text{ if } [(r+\alpha), n] = d, \quad d < n.$$

If $r + \alpha \equiv 0 \pmod{n}$, then $\epsilon^{-\beta(r+\alpha)} = 1$ for all values of β.

We have now shown that the coefficients of $X_{\frac{\gamma}{n_2}}$ in equation (29) vanishes except when $r + \alpha \equiv 0 \pmod{n}$. Hence we may write

$$T: \quad \sigma'_{\frac{\frac{n}{2}}{n}, \frac{\frac{n}{2}-\alpha}{n}}(nu) = \frac{\frac{1+i}{\sqrt{2}}}{i\sqrt[4]{\Delta}} \sum_{\gamma} X_{\frac{\gamma}{n^2}}(u). \tag{30}$$

where γ satisfies the congruence

$$\gamma + \alpha \equiv 0 \pmod{n}. \tag{b}$$

Now there are just n quantities less than n^2 satisfying this congruence, and since α is less than n, and, therefore, not divisible by n, neither is γ. Hence, in the summation on the left of equation (29), there are n quantities, $X_{\frac{\gamma}{n^2}}(u)$, not one of which is found in summation (28). Giving α all values from 0 to $n-1$, we get n equations, each containing n quantities, $X_{\frac{\gamma}{n^2}}(u)$. Moreover, an $X_{\frac{\gamma}{n^2}}(u)$ occurring in one equation derived from (29), by giving α all values, occurs in no other, for if so,

$$X_{\frac{\gamma}{n^2}}(u) = X_{\frac{\gamma'}{n^2}}(u).$$

Whence $\gamma + \alpha \equiv \gamma' + \alpha \pmod{n^2}$. That is, $\gamma - \gamma' + \alpha - \alpha' \equiv 0 \pmod{n}$, which is impossible. We now apply S to one of the equations obtained by giving α all values in (30):

$$S^k: \quad \sigma'_{\frac{\frac{n}{2}}{n}, \frac{\frac{n}{2}-\alpha}{n}}(nu) = \frac{\frac{1+i}{\sqrt{2}}}{i^{k+1}\sqrt[4]{\Delta}} \sum \epsilon'^{\left(\frac{n^2}{8} + \frac{\gamma^2}{2}\right)k} X_{\frac{\gamma}{n^2}}(u). \tag{31}$$

Let us now enquire for a value of k such that by an application of S^k to the summation on the left side of one of the equations (29), produces the same summation multiplied by a constant.

The evident conditions are that any $X_{\frac{\gamma}{n^2}}(u)$ that occurs in the given summation also occurs in the resulting summations, and that any two quantities,

$X_{\frac{\gamma}{n^2}}(u)$, $X_{\frac{\gamma}{n^2}}(u)$, $X_{\frac{\gamma'}{n^2}}(u)$ shall have the same coefficient. The last condition is equivalent to

$$\epsilon'\left(\frac{n^2}{8}+\frac{\gamma^2}{2}\right)^k=\epsilon'\left(\frac{n^2}{8}+\frac{\gamma'^2}{2}\right)^k.$$

We may now show that this is equivalent to

$$k\equiv 0 \pmod{n}.$$

Hence, giving k in equation (31) all values, we can derive n, and only n, different equations.

Hence, the values of A_a in equation (27) may be obtained by giving to a all values from 0 to $n-1$, in equation (29), and to k all values from 0 to $n-1$ in equation (31). We shall, at the same time, arrange these n^2 equations in n sets, each set containing n equations, and in every equation of each set, the left side is a linear homogeneous expression of the same n quantities, $X_{\frac{\gamma}{n^2}}(u)$, while some $\sigma_{\frac{\lambda}{n},\frac{\mu}{n}}(nu)$ constitutes the right side.

BLOOMINGTON, INDIANA.

Minors of Axi-symmetric Determinants.

By E. J. Nanson.

The linear relations between certain minors of any axi-symmetric determinant which are due to Kronecker, and certain extensions thereof due to Metzler and Muir,* are connected in a remarkably simple way with certain general determinant theorems due to Schweins and Sylvester. It is the object of this paper to establish the connection referred to and also to prove a general theorem which includes the whole of the relations mentioned as particular cases.

1. It has been shown long ago by Clebsch that any axi-symmetric determinant $|a_{pq}|$, where $p, q = 1, 2, \ldots n$, which, in Sylvester's umbral notation, is denoted by

$$\begin{pmatrix} 1 & 2 & 3 \ldots\ldots n \\ 1 & 2 & 3 \ldots\ldots n \end{pmatrix},$$

may be represented symbolically by

$$\frac{1}{n!}\,(abc\,\ldots\ldots)^2,$$

where

$$a_{pq} = a_p\,a_q = b_p\,b_q = c_p\,c_q = \text{etc.},$$

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* Phil. Mag., April, 1902, pp. 410-416.

and $(abc \ldots .)$ is an abbreviation for the determinant of order n whose r^{th} column is

$$a_r, \; b_r, \; c_r, \text{ etc.},$$

and the symbols a_p, b_p, etc., obey the fundamental laws of algebra, so that

$$a_{pq} = a_p a_q = a_q a_p = a_{qp},$$

as required on account of the symmetry assumed; and, consequently, a general determinant cannot be represented symbolically by Clebsch's method. Nevertheless, any minor of an axi-symmetric determinant can be so represented. Thus the minor formed with the r rows $\alpha, \beta, \gamma, \ldots . \varkappa$ and the r columns $\alpha', \beta', \gamma', \ldots . \varkappa'$ of $|a_{pq}|$ is represented in Sylvester's umbral notations by

$$\begin{pmatrix} \alpha & \beta & \gamma & \ldots . & \varkappa \\ \alpha' & \beta' & \gamma' & \ldots . & \varkappa' \end{pmatrix}$$

and may be represented in Clebsch's symbolic notation by

$$\frac{1}{r!} (a_\alpha b_\beta c_\gamma \ldots . k_\varkappa)(a_{\alpha'} b_{\beta'} c_{\gamma'} \ldots . k_{\varkappa'}),$$

where $(a_\alpha b_\beta c_\gamma \ldots . k_\varkappa)$ is an abbreviation for the determinant of order r formed with the r columns $\alpha, \beta, \gamma, \ldots . \varkappa$ of the array whose p^{th} column is

$$a_p, \; b_p, \; \ldots . \; k_p,$$

where there are r letters $a, b, \ldots . k$.

2. As shown by Clebsch, calculations involving the elements a_{pq} may be carried out by means of the symbols a_p, b_p, etc., provided care is taken not to introduce any power of any of these symbols higher than the second. It follows, therefore, that from any homogeneous quadratic relation connecting determinants of the type

$$(a_\alpha b_\beta c_\gamma \ldots . k_\varkappa).$$

where there are r letters $a, b, c, \ldots. k$, and the letters $\alpha, \beta, \gamma, \ldots. \varkappa$ may be any r numbers selected from $1, 2, 3, \ldots. n$, we can at once deduce a linear relation connecting minors of order r of any axi-symmetric determinant.

3. To obtain a quadratic relation of the kind mentioned, let $\alpha, \beta, \gamma, \ldots \varkappa$, $\alpha', \beta', \gamma', \ldots. \varkappa'$ be any two sets, each consisting of r numbers, chosen from $1, 2, 3, \ldots. n$. Let Δ denote the array whose p^{th} column is

$$a_p, b_p, c_p, \ldots. k_p,$$

where there are r letters $a, b, c, \ldots. k$. Also, let λ, μ, etc., be sets chosen from $1, 2, 3, \ldots. n$, and let the arrays, square or rectangular as the case may be, formed with the sets of columns λ, μ, etc., of Δ, be denoted for brevity by $(\lambda), (\mu)$, etc. Now, specializing, let λ, μ be complementary sets chosen from $\alpha, \beta, \gamma, \ldots \varkappa$, and let λ', μ' be complementary sets chosen from $\alpha', \beta', \gamma', \ldots \varkappa'$. We have then to consider the determinant

$$\begin{vmatrix} (\lambda) & (\mu) & (\lambda') & (0) \\ (0) & (\mu) & (\lambda') & (\mu') \end{vmatrix} \tag{1}$$

of order $2r$, where (0) denotes an array of zeros, the context showing in each case the dimensions of the zero array. Thus, in the first symbolic row the symbol (0) has as many rows as each of the horizontally collinear arrays $(\lambda), (\mu), (\lambda')$, that is, it has r rows, and it has as many columns as the vertically collinear array (μ').

4. Expanding the determinant (1) by Laplace's theorem in terms of the first r rows, the first term of the expansion is the product of the two r^{th} order determinants

$$|(\lambda)\ (\mu)|, \quad |(\lambda')\ (\mu')|.$$

It will be convenient to denote this product by S or Σ. Any other term in the Laplace expansion may be derived from S or Σ by interchanging one or more columns of the array (λ') with one or more columns of the array (μ) and chang-

ing the sign of the result when the number of columns interchanged is odd. If s, s' are the numbers of columns in the arrays (λ), (λ'), and, consequently, $r-s$, $r-s'$ are the numbers of columns in the arrays (μ), and (μ'), then the number of terms which can be derived in this way from S or Σ is

$$(r-s)_1 s'_1 + (r-s)_2 s'_2 + \ldots . + (r-s)_t s'_t,$$

where t is the smaller of the two, $r-s$, s', and $(r-s)_n$, s'_n denote the numbers of combinations of $r-s$, s' things n together. If the sum of all terms so derived from S or Σ be denoted by ${}_\mu\Sigma_{\lambda'}$, the expansion of the determinant (1) is

$$\Sigma + {}_\mu\Sigma_{\lambda'}.$$

5. To find another equivalent for the determinant (1), subtract rows $1, 2, \ldots . r$ from rows $r+1, r+2, \ldots . 2r$ respectively. Thus (1) becomes

$$\begin{vmatrix} (\lambda) & (\mu) & (\lambda') & (0) \\ (-\lambda) & (0) & (0) & (\mu') \end{vmatrix}, \tag{2}$$

where $(-\lambda)$ denotes the array formed from the array (λ) by changing the sign of every element.

If, now, the set λ' be more numerous than the set λ, the determinant (2) obviously vanishes, for the array represented by the symbols (0), (0) in the second symbolical row consists of r rows and of more than r columns.

If the two sets λ, λ' are equally numerous, then, by s interchanges of columns, and by s changes of sign, the determinant (2) takes the form

$$\begin{vmatrix} (\lambda') & (\mu) & (-\lambda) & (0) \\ (0) & (0) & (\lambda) & (\mu') \end{vmatrix},$$

and, therefore, has the value

$$|(\lambda')\ (\mu)| \cdot |(\lambda)\ (\mu')|,$$

which may be derived from S or Σ by interchanging λ, λ'.

Finally, if the set λ be more numerous than the set λ', it will be seen that the determinant (2) is the sum of the results of interchanging in S or Σ the set λ' with every equally numerous set contained in λ.

6. Denoting the sum of these results by ${}_{\lambda}S_{\lambda'}$, and equating the two expansions of the determinant (1), we have

$$\Sigma + {}_{\mu}\Sigma_{\lambda'} = {}_{\lambda}S_{\lambda'},$$

where ${}_{\lambda}S_{\lambda'}$ has the value zero, when the set λ' is more numerous than the set λ.

The result which has been obtained may be stated as follows:

If, in the product S or Σ of any two determinants

$$(\alpha\beta\gamma \ldots\ldots \varkappa), \quad (\alpha'\beta'\gamma' \ldots\ldots \varkappa')$$

of the same order, we divide the columns $\alpha, \beta, \gamma, \ldots\ldots \varkappa$ into two complementary sets λ, μ, and if we also divide the columns $\alpha', \beta', \gamma' \ldots\ldots \varkappa'$ into two complementary sets λ', μ', then denoting by ${}_{\lambda}S_{\lambda'}$ the sum of the results of interchanging the set λ', with every equally numerous set from λ and denoting by ${}_{\mu}\Sigma_{\lambda'}$ the sum of the results of interchanging one or more of the set λ' with one or more of the set μ, subject to the condition, in the latter case, that when the number of columns interchanged is odd, the sign of the determinant product is changed, we have

$$S = \Sigma = {}_{\lambda}S_{\lambda'} - {}_{\mu}\Sigma_{\lambda'}.$$

7. From this theorem it immediately follows that

If K or M be any minor

$$\begin{pmatrix} \alpha & \beta & \gamma \ldots\ldots \varkappa \\ \alpha' & \beta' & \gamma' \ldots\ldots \varkappa' \end{pmatrix}$$

of an axi-symmetric determinant of odd or even order, and if the row suffixes $\alpha, \beta, \gamma, \ldots\ldots \varkappa$ be divided into two complementary sets λ, μ, and also the column

suffixes α', β', γ', ... $\varkappa'$ *into two complementary sets* λ', μ', *then*

$$K = M = {}_{\lambda}K_{\lambda'} - {}_{\mu}M_{\lambda'},$$

where ${}_{\lambda}K_{\lambda'}$ *denotes the sum of the results of interchanging in* K *or* M *the set* λ' *with every equally numerous set from* λ, *and* ${}_{\mu}M_{\lambda'}$ *denotes the sum of the results of interchanging in* K *or* M *one or more of the set* λ' *with one or more of the set* μ *and changing the sign of the minor when the number of suffixes interchanged is odd.*

8. The expression ${}_{\lambda}K_{\lambda'}$ may be called a Kronecker expression and the expression $K - {}_{\lambda}K_{\lambda'}$ may be called a Kronecker aggregate. Also, the expression ${}_{\mu}M_{\lambda'}$ may be called a Metzler expression and the expression $M + {}_{\mu}M_{\lambda'}$ may be called a Metzler aggregate. It is also convenient to speak of ${}_{\mu}M_{\lambda'}$ as the Metzler expression complementary to the Kronecker expression ${}_{\lambda}K_{\lambda'}$. We then have the theorem that

Any minor of an axi-symmetric determinant is equal to a Kronecker expression diminished by the complementary Metzler expression.

9. In order to compare this theorem with results previously known, suppose that λ, μ, λ', μ' consist of s, t, s', t' suffixes respectively, so that

$$s + t = s' + t' = r,$$

where r is the order of the minor. First, let $t = 0$, then the Metzler expression vanishes, and we have the theorem that any minor of an axi-symmetric determinant is equal to the sum of the minors derived from it by interchanging a fixed set of column suffixes with every equally numerous set of row suffixes. This result may also be expressed by saying that a Kronecker aggregate vanishes when *all* the row suffixes are subject to interchange, and was published by the present writer in the Messenger of Mathematics for January, 1902. When $s' = 1$, that is, when the fixed set of row suffixes consists of a single suffix, the result reduces to the linear relation originally given by Kronecker.

10. Next, let $s' > s$, then the Kronecker aggregate vanishes, and we have the theorem that any minor of an axi-symmetric determinant is equal to the sum of the minors derived from it by interchanging one or more of a fixed set of column suffixes with one or more of a fixed set of row suffixes, provided, first that the sign of every derived minor is changed when the number of suffixes interchanged is even, and provided, second, that the total number of suffixes subject to interchange exceeds the order of the minors involved. This result may also be expressed by saying that a Metzler aggregate vanishes whenever the total number of suffixes subject to interchange exceeds the order of the minors involved. Of this theorem, two special cases have hitherto been given. First, when all the row suffixes are subject to interchange, a Metzler aggregate, as here defined, is identical with a Metzler aggregate as defined by Muir, l. c., and vanishes, as shown by Metzler. Second, when the total number of suffixes subject to interchange exceeds the order of the minors involved by one unit, then a Metzler aggregate, as here defined, is identical with a Metzler sub-aggregate as defined by Muir, and vanishes as stated by Muir, l. c., p. 416. Finally, when $s = r - 1$, $s' = r$, the Metzler aggregate reduces to the vanishing aggregate originally given by Kronecker.

11. When $s = s'$ and, consequently, $t = t'$, the general theorem takes an interesting form. The Kronecker expression then reduces to a single term, viz. the determinant obtained by interchanging the sets λ, λ'. This single term is, therefore, equal to a Metzler aggregate, in which the total number of interchangeable suffixes is equal to the order of the minors involved. This result agrees with those indicated by Muir, l. c., p. 415, and may be expressed in the form

$$\begin{pmatrix} \lambda' & \mu \\ \lambda & \mu' \end{pmatrix} = \Sigma \begin{pmatrix} \lambda & \bar{\mu} \\ \underline{\lambda'} & \mu' \end{pmatrix},$$

where the first member denotes the minor formed with the rows λ', μ and the columns λ, μ' of an axi-symmetric determinant, the sets λ, λ' and, consequently, also the sets μ, μ' being equally numerous. The first term in the second member is the minor formed from the first member by interchanging the sets λ, λ', and the rest of the terms are derived from the first term by interchanging, as indicated by the horizontal bars used by Muir, one or more of the set λ' with

one or more of the sets μ, the sign being changed whenever the number of interchanges is odd. But the theorem thus obtained is readily seen to be at once derivable from the theorem already mentioned as having been given in the Messenger of Mathematics. For, if we start with the minor $\begin{pmatrix} \lambda' & \mu \\ \lambda & \mu' \end{pmatrix}$, then the Kronecker expression derived therefrom by interchanging the fixed set λ' with every equally numerous set which can be formed from the combined set λ, μ' is obviously identical with the expression $\Sigma \begin{pmatrix} \lambda & \bar{\mu} \\ \underline{\lambda'} & \mu' \end{pmatrix}$ already defined.

MELBOURNE, *August*, 1903.

On the Forms of Sextic Scrolls having a Rectilinear Directrix.

By Virgil Snyder.

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1. In Volume XXV of the Journal I have three papers on sextic scrolls, pp. 59–84, 85–96, 261–268, wherein 118 types of the surface are discussed and enumerated. Besides the references there given, three papers were in existence which treat exclusively or partly of sextic scrolls. The first of these is the doctor dissertation of Dr. Karl Fink, "Ueber windschiefe Flächen im allgemeinen und insbesondere über solche des sechsten Grades," Tübingen, 1886. The first few pages are devoted to a general discussion of the correspondence between the points of any two plane sections of scrolls, including the formulas which I have given on p. 75 and on p. 268, but various false conclusions are drawn from the results. The principles thus established are applied to the S_6, but so carelessly that over half the types mentioned are impossible, and a large number of others exist of which no mention is made. The paper is practically worthless.

The second paper is the doctor dissertation of Dr. Jakob Bergstedt, "Om regelytor af sjette graden, I, unikursale ytor,'' Lund., 1886. The abstract of this memoir in the Fortschritte der Mathematik is rather misleading. By no means all of the forms of unicursal sextics are derived, yet the number is over 60, the equations of many being derived. The paper has just about the same degree of completeness as my first paper, and nearly the same methods are employed, except that Bergstedt more systematically analyzes the configurations of the multiple points on the nodal curve and uses correspondence but a few times. Several false conclusions are arrived at. These two papers would not materially affect the truth of the statement that no (complete) systematic dis-

cussion on S_6 exists. The third paper, however, the doctor dissertation of Professor Anders Wiman, "Klassifikation of regelytorna af sjette graden," Lund., 1892, is of very different nature. The memoir is not even mentioned in the Fortschritte, yet it contains 111 pages and enumerates 118 types of the surface. The method employed by Wiman is radically different from any of the preceding. He establishes a (1, 1) correspondence between the points of space and the lines of a complex, by means of which the surface becomes a twisted curve. But few equations are given, the existence of the various types being established by geometric methods. In some cases the equations can be easily obtained, while in others the difficulty is shifted to the twisted curve, satisfying prescribed conditions.

In the present paper I wish to complete the enumeration, including the derivation of the equation, of those scrolls which have a rectilinear directrix, and to correct the errors in my previous results. The same methods will be employed as were used in the preceding papers. The purpose is to complete Wiman's results by deriving the equations, and to add those which he overlooked.

A.—*Unicursal Scrolls.*

§1. *Simple directrix line.*

2. As was shown, p. 76 (of my own paper), type XXX, a S_6 with a simple directrix line c_1^1, and c_{10}^2 with a multiple point, through which pass 4 generators, exists. This point is a sixfold point on c_{10}, which, consequently, lies on a cone of order 4, k_4. B., p. 19, proves that k_4 is of genus $p = 3$. Suppose, now, c_1 pierces the plane of c_5 in a point on a double tangent. The points of intersection of the tangent and c_5 are consecutive, hence the plane contains two pairs of torsal generators and a simple generator. The former intersect in a double point on c_{10}^2, the tangents at which are the torsal generators. Similarly, c_1^1 may lie on two or three double tangents, giving rise to two or three further double points on c_{10}, the latter being unicursal.

Finally, if the point of intersection be on four double tangents, the nodal curve will be composite. Let c_5 be defined by

$$(x^4 + y^4)(y + ax) - (y^2 + yz + bz^2)(y + dz) = 0, \quad w = 0,$$

wherein

$$a - d - 2 = 0. \quad 2b + 2d + 1 = 0, \quad 2b + d + 2bd = 0,$$

and let c_1 be $x = 0$, $y = 0$. The twisted quintics will each have a triple point at (1, 0, 0, 0) and will intersect in four other points. Each curve will lie on a quadric cone. When four generators are concurrent, the only other case is that of a fivefold line, which was exhaustively discussed in types I to XII of my first paper.

3. Now establish a similar (1, 1) correspondence between c_1 and c_5, the latter being unicursal and having a triple point. The resulting c_{10}^3 on S_6 has four triple points and is of genus 3 (B., p. 21). Here again 1, 2, 3 or 4 double points may appear, the last case necessitating that c_{10} break up into two c_5, each having two double points through which the other curve passes singly. Let A be one of these points. The cones, having A for vertices, are of order 3 for c_5 and of order 4 for c_5'. The former has one double generator which is a single generator of the latter, and the latter has three double generators, two of which are single on the former. The cones still have six common generators. Consider the plane passed through c_1 and A. It will contain five generators to S_6, three of which pass through A. Each generator must contain two points on each c_5. From a figure, it is easily seen that two of these generators of S_6 are common generators to the two cones, while the third is a double generator to k_4. This accounts for two further intersections of the cones, hence c_5 and c_5' have four actual or apparent points of intersection. It was seen above that they have four actual intersections, at all of which the tangents to each intersect c_1. A simple directrix may also be a simple or a multiple generator, but as the order of the residual nodal curve is correspondingly lower, such cases will be considered in connection with multiple directrices. Wiman does not distinguish between them.

§2.—*Double directrix line.* (a). *No double generators.*

4. Besides XIX, in which c_9 is of genus 1, another form exists by letting c_1 lie on a double tangent, as another double point appears. If c_1 lies on two double tangents, c_9 breaks up into c_6 having a fourfold point and a plane c_3 having its double point at the same point. The two curves have two points of intersection. If c_1 lies on three double tangents, c_6 breaks up into two plane

cubics having a common point for node. Each pair intersect in one further point.

5. By establishing a (1, 2) correspondence between a c_5 with a triple point and a c_1 which passes through a simple point as self-corresponding point, the S_6 has a c_9 with four triple points and of genus 1. The double point may appear as above. If two double points appear, c_9 breaks up into a c_6 with four double points and a twisted c_3 passing through each double point and two simple points on c_6. c_1 cuts c_3 once and c_6 twice. When c_1 lies on three double tangents, c_6 breaks up into two c_3 which pass through the four nodes, intersect each other in one point and each intersects the first c_3. Each c_3 cuts c_1 once. If, instead of a (1, 2) correspondence between c_1 and c_5 with a self-corresponding point at their intersection, we establish a general (1, 1) correspondence, these eight forms appear again, except that c_1 is now simple directrix and simple generator. Four of these were given in forms XXXI–XXXIV.

(b). *One Double Generator.*

6. If, in the (1, 2) correspondence between c_5 and c_1, the two values of the parameter on a node should correspond to the point on c_1, the line joining them is a double generator g^2. The six types analogous to (a) are found, but the symbol $c_1^2 + c_6^2 + c_3^2$ can break up into $c_1^2 + c_5^2 + c_3^2 + g^2$ or $c_1^2 + c_6^2 + c_2^2 + g^2$ in both systems, those arising from a plane quintic directrix having a fourfold point and those arising from a plane unicursal quintic directrix having a triple point. By accounting for a sixfold point and the required number of nodes in the first case and four triple points and appropriate nodes in the second, the lines have the following configurations: $c_1^2 + c_8^2 + g^2$. The c_8 has a fivefold point. The g^2 cuts c_8 in the fivefold point and one simple point. c_1 cuts c_8 in two points $c_1^2 + c_6^2 + c_2^2 + g^2$.

c_6 has a fourfold point, passing through g^2. c_2^2 passes through the fourfold point and cuts g^2 in one other point and c_6 in two simple points, c_1 cuts c_6 twice. This is derived from type LXIII, eq., p. 90, by making g^2 in plane of c_2. $c_1^2 + c_5^2 + [c_3^2] + g^2$. c_5 has a triple point at node of c_3, the two curves having two points of simple intersection. g^2 passes through the multiple point and one

other point of c_5. c_1 cuts each curve once. Let

$$x=1-\lambda^2,\quad y=\lambda(1-\lambda^2),\quad z=0,\quad w=1 \text{ be } c_3,$$
$$x=1,\quad y=0,\quad z=0,\quad w=1 \text{ be } c_1,$$

and let λ, μ be connected by a correspondence of the form

$$\lambda^2\left(4a(a+c)\mu^2+4a'(a+c)\mu+a'^2\right)+4c(a+c)\mu^2=0.$$

The resulting S_6 is of the form required.

For the case $c_1^2+c_2^3+2\,[c_3^2]+g^3$ there is no ambiguity, and the equation is derived directly from the general case.

7. In the second case, the c_8 has two triple points and two double points, g^3 passing through both of the latter and one simple point. c_8 cuts c_1 in two points, and is of genus one. One double point may appear when c_1 lies on a double tangent. For the case $c_1^2+c_6^2+c_2^3+g^2$, c_6 has two double points through which g^3 passes, and two others, through which passes c_2. The latter cuts c_6 in two simple points and g^2 once. c_1 cuts c_2 twice. The two cases of the same symbol may be easily distinguished by the position of g^3. In the [2, 4] case, it lies in the plane of c_2, while in the [3, 3] case it does not. The symbol $c_1^2+c_5^2+c_3^2+g^2$ differs from the preceding by having a twisted c_3. The c_5 has three simple points on g^2, it also has two double points, through each of which c_3 passes. c_5 and c_3 each cut c_1 in one point $c_1^2+2c_3^2+c_2^3+g^2$; both of the c_3 are twisted. They intersect in four points, through two of which passes c_2 and through the other two g^2. The two c_3 intersect in one other point; c_2 intersects each c_3 and g^2 in one point besides the common intersection. c_1 cuts each c_3 once.

(c). *Two Distinct Double Generators.*

8. In the same manner, as in (b), introduce two double generators, intersecting in the fourfold point of c_5. The residual is a c_7^2 having a fourfold point at the intersection of the double generators, and one point on each g^3 as well as on c_1, $p=1$, and can be made zero in the usual way.

If the two nodes on c_5 are consecutive, the $2g^3$ can become tacnodal or consecutive. This scroll can also be generated as follows: Given a tacnodal c_4 with

one other double point. Take any line cutting the tacnodal tangent as directrix in a (1, 2) correspondence with c_4, the point of intersection being a pinch-point. The tacnodal tangent will become a tacnodal generator. The equations of c_4 may be

$$w=0, \quad (yz-mx^2)(yz-m'x^2)=fx^3y+gx^2y^2.$$

The point (0, 0, 1, 0) is the tacnode, with $y=0$, $w=0$, for tangent (0, 1, 0, 0) is a crunode. Let c_1 be $y=0$, $z=0$, $w=k$, $x=1$. It is projective with the pencil $x=kz$, $w=0$. Connect the two points in which a line of this pencil cuts c_4 with the corresponding point on c_1. The S_6 is the k eliminant of $k^2z=kx-w$ and

$$[y-m(kx-w)][y-m'(kx-w)]=fy^2k^3+gk^2y^2.$$

The plane $y=0$ contains the double directrix and the generator $w=0$ as a four-fold line. $p=1$ or 0.

9. Similarly for the [3, 3] case. c_7 now has four double points, two of which lie on each g^2. Each g^2 cuts c_7 in one simple point. $p=1$ or 0. In case of the tacnodal generators, c_7 has two triple points. It intersects $\overline{2g^2}$ in four points, two of them having their tangents in the plane of the singular g^2. If, in the [2, 4] case, c_1 lies on two double tangents, the symbol becomes $c_1^2+c_4^2+[c_3^2]+2g^2$, the double generators being distinct. c_4 has a node in common with $[c_3]$ and two simple points in common with it. c_3 cuts c_1 once. No further forms can exist when the double generators are distinct. When they are consecutive, the first new type becomes $c_1^2+c_5^2+c_2^2+\overline{2g^2}$. The S_6 may be generated as follows:

Let c_2 be $\qquad w=\lambda, \quad x=\lambda^2, \quad y=1, \quad z=0,$
and c_1 be $\qquad x=0, \quad y=0, \quad z=\mu, \quad w=1.$

If $\lambda=0$, $\mu=0$ be a double element in the (2, 2) correspondence between λ, μ, the line $x=0$, $z=0$ will be a tacnodal generator. c_5 has a triple point, c_2 touching one branch, with g^2 as tangent. c_5 and c_2 have one other point in common. Finally, if c_1 lies on three double tangents, the symbol becomes $c_1^2+[c_3^2]+2c_2^2+\overline{2g^2}$. Consider the conics:

$$c_2, \quad x=1, \quad y=\mu, \quad z=0, \quad w=\mu^2,$$
$$c_2', \quad x=0, \quad y=\lambda, \quad z=1, \quad w=\lambda^2,$$

which touch each other at (0, 0, 0, 1) with $x=0$, $z=0$ for common tangent. The equations of the line joining λ to μ are

$$\mu x+\lambda z-y=0, \quad \mu^2 x+\lambda^2 z-w=0.$$

Any line cutting the common tangent is of the form

$$bx-z=0, \quad cx+y+aw=0.$$

A generator will cut this line when

$$ab\lambda^2+a\mu^2+b\lambda+\mu+c=0.$$

The equation of S_6 becomes

$$\begin{vmatrix} a(z-bx) & z-bx & abw+by+cz & 0 \\ 0 & a(z-bx) & z-bx & abw+by+cz \\ x(x+z) & -2xy & y^2-zw & 0 \\ 0 & x(x+z) & -2xy & y^2-zw \end{vmatrix}=0.$$

The section made by the plane $bx-z=0$ is $x^4(aw+by+cx)^2$. The residual nodal curve is of order 3 and cuts every generator once. It cuts c_1 once. The plane $bx-z=0$ cuts c_1 and $x=0$, $z=0$, hence the latter cuts c_3 twice. These points must coincide at (0, 0, 0, 1), since $x=0$, $y^2-zw=0$ define the complete intersection with $z=0$. Hence c_3 has a node at (0, 0, 0, 1). Any plane section through the singular generator will contain a quartic having one double point, and a tacnode at (0, 0, 0, 1).

If the (2, 2) correspondence of the preceding case has a cusp at (0, 0), the result will have this symbol, except that c_3^2 has a cusp at the point of contact. An illustration is afforded by the surface

$$[x(z^2+w^2+xy)+y(z^2-2xz)]^2-4xyw^2(x-z)^2=0,$$

which has a nodal cubic in the plane $w=0$. The c_1^2 is $x=0$, $y=0$, $\overline{2g^2}$ is $x=0$, $z=0$. One double conic is $xy-w^2=0$, $z=0$.

10. If, in the [3, 3] case, c_1 lies on two double tangents, the symbol may be either $c_1^2+c_4^2+c_3^2+2g^2$ or $c_1^2+c_5^2+c_2^2+2g^2$, according as the second g^2 factors off the c_5 or the c_3.

If, between c_3, c_1 as defined in §6, the correspondence be of the form

$$(a+b)\lambda^2\mu^2 = a\lambda^2 + b\mu^2,$$

the residual is a c_4 having three points on each g^2. c_4 cuts c_3 twice. The difference between these forms is, here c_3 is twisted. If the two points on c_2 correspond to the point of intersection of c_1, with its plane in a (2, 2) correspondence, and a double element exists, as

$$(a\lambda^2 + b)\mu^2 + (\lambda^2 + 4bc)(\mu + c) = 0,$$

the two generators will be distinct. The double element is $\lambda = 0$, $\mu = -2c$, while the branch-point is $\mu = 0$, $\lambda = \pm\sqrt{-4bc}$.

The residual curve is c_5 having two double points on one g^2 and three simple points on the other. It cuts c_3 four times, c_2 passing through two intersections of c_5 and g^2.

In case of three double tangents, the symbol becomes $c_1^2 + c_3^2 + 2c_2^2 + 2g^2$. c_3 has two points on each (distinct) g^2 and one on c_1. Each c_2 cuts c_3 in two points on one g^2 and cuts the other g^2 once. The two c_2 have one point in common.

Only two specializations of the tacnodal g^2 can occur; $c_1^2 + c_5^2 + c_2^2 + \overline{2g^2}$. Given $x = \lambda^2$, $y = 1$, $w = \lambda$, $z = 0$; $x = 0$, $y = 0$, $z = \mu$, $w = 1$. The tangent to c_2 at $\lambda = 0$ cuts c_1. The plane containing both is $y = 0$. If the (2, 2) correspondence between λ, μ have $\lambda = 0$, $\mu = \mu_1 \neq 0$ for double element, the line joining 0 to μ is a tacnodal generator not in the plane of the c_2. The c_5 has three points on $\overline{2g^2}$, at two of which it touches the singular plane $y = 0$. Finally, for three double tangents, let $\mu x + \lambda z - \lambda\mu w = 0$, $\lambda x - y + \mu z = 0$ be connected by $\lambda(\mu - 1)^2 + \alpha\mu(\lambda - 1)^2 + \beta(\lambda^2 - \mu^2) = 0$. The plane of the tacnodal g^2 is $2(x + z) - w - y = 0$. This procedure proves that the types LV, LIX and LXI of my enumeration, p. 84, are impossible. In the first case, a surface of type LV must be elliptic and c_7^2 should be replaced by c_5^2. Similarly for LXI, but LIX is interesting from the fact that a S_6 having $2c_2^2$ and c_1^2 must have two double generators to be unicursal. The same result can also be reached by Cayley's method. The method of correspondence cannot be relied upon as a sufficient ground of classification, without interpreting each step geometrically.

Thus, types LV and LXI can be proved impossible by the paper in which they are enumerated, but LIX cannot be thus (directly) explained.

(d). *Three Double Generators.*

11. Three pairs of nodes can only correspond to double generators when one of them is tacnodal. In the case of the plane c_5 has a fourfold point; this requires that all three double generators will be consecutive. The residual curve is a unicursal c_6^2 having a triple point on the singular generator. Any plane section will have an oscnode on the singular generator.

The scroll may be generated as follows:

Given the oscnodal quartic

$$(yw - x^2)^2 = y^3(w - y), \quad z = 0 .$$

The line

$$y = kx$$

cuts the quartic in two points distinct from the node. Let this pencil be projective with the range

$$x = 0, \quad y = y, \quad z = k, \quad w = 0.$$

Connect the points of the range with the points in which the corresponding line cuts the quartic. Let the points be denoted by $(x_1, y_1, z_1, 0)$.

$$(y_1w_1 - x_1^2)^2 = y_1^3(w_1 - y_1),$$
$$y_1 = kx_1,$$
$$xw_1 = x_1w, \quad y_1xk = x_1(ky - z),$$

hence,

$$k^4x^2 - k^3xw + k^2w^2 - 2kwx + x^2 = 0,$$
$$k^2x - ky + z = 0.$$

The resultant is of the form

$$\begin{vmatrix} 1 & -w & w^2 & -2wx & x^2 & 0 \\ 0 & x & -xw & w^2 & -2wx & x^2 \\ 1 & -y & xz & 0 & 0 & 0 \\ 0 & +1 & -y & z & 0 & 0 \\ 0 & 0 & x & -y & z & 0 \\ 0 & 0 & 0 & x & -y & z \end{vmatrix} = 0.$$

The surface is not contained in a linear congruence. In the [3, 3] case, the singular line is distinct from the ordinary double generator. The residual curve is a c_6^2 having two double points and one simple point on g^2, and four simple points on the singular $\overline{2g^2}$, being touched by the singular torsal plane in two of them. As any plane through c_1^2 contains four generators, c_6^2 does not intersect c_1^2.

If, in the correspondence of §9, the two points in which $z = 0$, $y = mx$ cuts the conic, both correspond to $\mu = 0$, while each point has $\mu = 0$ for pinch-point, the joining line is an ordinary double generator. The residual curve is a rational c_4 having three points on the g^2. To make three consecutive generators, the c_5 should have three coincident cusps. Let

$$x^5 - yw^4 = 0, \quad z = 0$$

be the equations of c_5, and let $x = 0$, $y = 0$ be the line c_1. In parameters

$$c_5 : x = \frac{1}{\lambda}, \quad y = \frac{1}{\lambda^5}, \quad z = 0, \quad w = 1,$$

$$c_1 : x = 0, \quad y = 0, \quad z = \mu, \quad w = 1.$$

The point of intersection of c_1, c_5 must be a simple self-corresponding point ($\mu = 0$, $\lambda = \infty$). The cusp, $\lambda = 0$, must be a double root ($\mu = \infty$, $(\lambda = 0)^2$), hence

$$\lambda^2\mu = \lambda - 1.$$

The equation of S_6 is

$$(x + z)^4x^2w^4 - xy(x + w)^4 + 4xyw(x + z)(x + w)^2 - 2xyw^2(x + z)^2 = 0.$$

$x = 0$, $y = 0$ is c_1^2. $x = 0$, $w = 0$ is three consecutive double generators, as any plane section will cut this line in an oscnode. In the plane $x + w = 0$ lies the double conic

$$(x + z)^2 + xy = 0,$$

which touches the singular generator.

The residual nodal curve is a twisted nodal quartic having at (0, 1, 0, 0) a node. The symbol is $c_1^2 + c_2^2 + c_4^2 + \overline{3g^2}$. Since c_1 intersects c_5 on inflexional tangent, no further forms can appear. Finally, when c_1 lies on two (consecutive) double tangents, the [3, 3] form becomes $c_1^2 + 3c_2^2 + 3g^2$, two g^2 being consecutive. The surface may also be generated as follows: Two conics cut each other

in two points P, Q, and the line joining P, Q is the ordinary g^3. The correspondence between the two conics is (2, 2), having P, Q as simple self-corresponding elements, with the second element at each corresponding to the other. The correspondence is determined by imposing the condition that the line joining corresponding points shall cut a fixed director c_1. Finally, the tacnodal generator must lie in a common tangent plane to the conics through c_1. The correspondence reduces to the form

$$\lambda\mu(\lambda+\mu)+4\lambda\mu+\lambda+\mu=0.$$

The c_1^2 has the equations

$$4z+y+w=0, \quad x=z$$

and the tacnodal generator is

$$x+z+w=0, \quad x+y+z=0.$$

The singular torsal plane is $2z+2x+y+w=0$, which also contains the third double c_2. The equation of S_6 becomes

$$\begin{vmatrix} z-x & -(y+4x) & 1 & x & 0 \\ 0 & z-x & 0 & y+w+4x & x-z \\ wx & x^3-z^3 & y & 0 & 0 \\ 0 & w & 1 & -z & 0 \\ 0 & 0 & z & x^2-wy & zy \end{vmatrix} = 0.$$

See Volume XXV, p. 80 ff. of the Journal. Another form of correspondence is $\lambda(\mu-1)^2+\alpha\mu(\lambda-1)^2=0$.

12. The quartic curve

$$(yz-x^2)^2=y^3(z-y), \quad w=0,$$

has an oscnode at $(0, 0, 1, 0)$, $y=0$ being the tangent. Let $y=mx$ cut the curve in two points x_1, z_1, .. such that

$$(mz_1-x_1)^2=m^3x_1(z_1=mx_1).$$

Make the lines of the pencil $y=mx$ projective with the range $y=0$, $z=0$, $w=mx$. Lines joining points on the range to x_1z_1 will be of the form

$$yz_1 = mx_1z, \quad y=(mx-w).$$

The equation of the surface becomes

$$(y+w)^4(z^2-yz+y^2)-2zx^2y(y+w)^2+x^4y^2=0.$$

The line $y=0, z=0$ is a double directrix, $y+w=0$, $x=0$ is a fourfold directrix. The line $y=0$, $w=0$ is an oscnodal or three consecutive generators. The surface has no other nodal lines.

The same configuration will appear whenever the tangent to the oscnode cuts the directrix. The tangent and point of correspondence do not need to be in united position.

Double Contact Directrix.

13. Given a c_5 such that the values of m corresponding to

$$y-\beta=m(x-\alpha),$$

α, β on c_5 form a quartic involution on x. Further, let the four values of x be arranged in two pairs, to the first of which corresponds μ_1, and to the second μ_2. Let m, μ be in (1, 2) correspondence and μ define a point on c_1, $x=\alpha$, $y=\beta$. Finally, if the tangent at α, β correspond to $\mu=0$ doubly, lines joining corresponding points will generate a S_6. The line c_1 will be double, but any plane section of S_6 not containing c_1 will have a tacnode at the point in which the latter pierces the plane. Any plane through c_1 will contain four generators, two of which pass through μ_1 and two through μ_2. Let α, β be 0, 0. c_5 may be defined by

$$x\,[\phi_2(x, w)]^2+y\,[\psi_2(x, w)]^2=0;$$
$$\phi_2(x, w)=x\,(ax+a'w); \quad \psi_2(x, w)=bx^2+b'xw+b''w^2.$$

It has a fourfold point at (0, 1, 0, 0). If

$$\mu^2=m, \quad y=mx,$$

then

$$[\phi_2(x_1, w_1)+\sqrt{m}\,\psi_2(x_1, w_1)][\phi_2(x_2, w_2)-\sqrt{m}\,\psi_2(x_2, w_2)]=0.$$

The form of the equation becomes

$$xy\left[ax^2 + a'xy - b'xz - 2b''zw\right]^2$$
$$= y^2\left[\psi_2(x,\, w)\right]^2 + 2(b'' - a')\, xyz^2\, \psi_2(x,\, w) + b''^2x^2z^4.$$

The two values of μ corresponding to it coincide, hence the line $x = 0$, $w = 0$ counts for three consecutive generators. When the two values of μ corresponding to $m = \infty$ do not coincide, there are two distinct double generators in the plane $x = 0$.

In general, corresponding to four double points on c_5 will be four intersections of a μ_1 generator with a μ_2 generator. If c_5 be unicursal, S_6 must, therefore, have $2g^2$. These may be in different planes or the same plane, and in either case, if the tangent cuts c_1, the generator will be singular, counting for two. When the generators are distinct, the residual is a c_6^2 which cuts c_1 twice. In the other case, it becomes a c_5 cutting c_1 once. There are, therefore, four types of unicursal forms in which the residual nodal curve does not degrade. When c_5 is of genus 1, the double generator may be ordinary or singular. All the forms in which the residual nodal curve is composite will be considered separately.

A particular form of this surface can be generated by the forms

$$\lambda^4x + y = 0, \quad \lambda^3w + \lambda y + z = 0.$$

When the double generators do not intersect, the surface can be generated from a c_5 having a triple point. The equations can most easily be obtained from the dual of a later form. Three types exist; two distinct double generators and c_6^2 having two points on the double directrix and two double points on each g^2; an ordinary and a tacnodal generator, the residual being a c_5^2 having two double points on g^2, one point on $\overline{2c_1^2}$ and a double point on $2g^2$ with one branch touching it; finally, two $\overline{2g^2}$ and a c_4^2, which has a singular osculating plane through each singular generator and the directrix. It may be generated by the two cones

$$\lambda^3x - \lambda^2w - z = 0, \quad \lambda^3w + \lambda z + y = 0.$$

The tacnodal directrix is $x = 0$, $y = 0$, and the singular double generators are $x = 0$, $w = 0$; $y = 0$, $z = 0$. Any plane will cut each of these generators in a tacnode. The equation of the scroll may be written

$$(xz^2 + yw^2)^2 + x^2y^2(4zw + xy) = 0.$$

Finally, c_4^3 may break up into $2c_2^3$. Given c_1: $x=0$, $w=0$, $z=\lambda y$, and c_2: $xw-y^2=0$, $z=0$. Pass a plane $x=mw$ through c_1, and let $m(\lambda+1)^2=(\lambda-1)^2$. The plane will cut c_2 in two points $(x_1, y_1, 0, w_1)$. From the equations of a line joining $(0, 1, \lambda, 0)$ to $(x_1, y_1, 0, w_1)$ we obtain

$$\lambda^2(y^2-xw)-2yz\lambda+z^2=0, \quad \lambda^2(x-w)+2\lambda(x+w)+x-w=0,$$

the λ eliminant of which is a S_6 having $2c_1^3$, $x=0$, $y-z=0$ and $w=0$, $y+z=0$ for tacnodal generators, and two double conics.

These forms are not mentioned by Wiman, but he notices the omission, p. 95.

14. Given the conic c_2, $xy-w^2=0$, $z=0$, and the straight line $c_1\, x=0$, $y=0$. Establish a (1, 2) correspondence between the planes π through c_1 and the points P_1, P_2 on c_1; let π cut c_2 in A and B. Connect A with P_1 and P_2, and B with P_1 and P_2. c_1 will be a double line and c_2 a double conic on the resulting S_6. The four lines lying in any plane π through c_1 have the peculiar property of meeting c_1 in pairs, hence the last statement made on p. 61, third paragraph, is incorrect.

Let the point $(0, 0, \mu, 1)$ on c_1 be joined to $(x_1, y_1, 0, w_1)$ on c_2.

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{w\mu-z}{\mu w_1}.$$

Let the plane π contain $(x_1, y_1, 0)$, so that

$$x_1=my_1.$$

Finally, let $m=\dfrac{a'\mu^2+b'\mu+c'}{a\mu^2+b\mu+c}$,

$$m=\frac{x}{y}, \qquad \mu\sqrt{my}=\mu w-z, \qquad \mu^2xy=\mu^2w^2-2\mu wz+z^2,$$

$$(ax-a'y)\mu^2+(bx-b'y)\mu+(cx-c'y)=0,$$

$$(w^2-xy)\mu^2-2wz\mu+z^2=0.$$

$$[(w^2-xy)(bx-by)+2wz(ax-a'y)][z^2(bx-b'y)+2wz(cx-c'y)] + [(cx-c'y)(w^2-xy)-z^2(ax-a'y)]^2=0.$$

The line $z = 0$, $cx - c'y = 0$ is a double generator, and c_1 is a tacnodal directrix. Since the (2, 2) correspondence between c_1 and c_3 has no double element, the S_6 is of genus 1. The residual nodal curve is a c_4 of the first kind cutting c_3 in the two points of intersection with g^2, and in the points (0, 1, 0, 1), (1, 0, 0, 1). The symbol is $\overline{2c_1^3} + c_3^2 + c_4^2 + g^2$ $(p = 1)$.

If $b' = c' = 0$, the line $x = 0$, $z = 0$ counts for a tacnodal generator. The symbol is $\overline{2c_1^3} + c_2^2 + [c_3^2] + 2g^2$ $(p = 1)$, the cubic being of genus 1. If $b'^2 = 4a'c'$, there is an ordinary double generator in the plane of the conic, and a singular tacnodal one in the plane $x = 0$. The symbol is $\overline{2c_1^3} + c_2^2 + [c_3^2] + \overline{2g^2} + g^2$ $(p = 0)$. Finally, if $b' = c' = a = 0$, the S_6 is unicursal and the singular line, which touches c_3^2 counts for three double generators. The residual nodal curve is a cubic having a double point at the point of contact of c_3 and g. The symbol is $\overline{2c_1^3} + c_2^2 + [c_3]^2 + \overline{3g^2}$ $(p = 0)$.

15. In the same manner, let a (1, 2) correspondence between the planes through c_1 and the points on c_1 be given, and let c_1 cut a twisted cubic c_3 in one point. Then will π cut c_3 in two points.

Let $x = \lambda$, $y = \lambda^2$, $z = \lambda^3$ be c_3, and let $z = 0$, $x = 0$ be the line.

$$x = mz, \qquad m = \frac{a'\mu^2 + b'\mu + c'}{a\mu^2 + b\mu + c}.$$

There are two positions of the plane such that the two points associated with one of the values of μ in this plane are collinear, hence the line joining them is a double generator. The symbol is $\overline{2c_1^3} + 2c_3^2 + 2g^2$, both cubics being space curves.

If the twisted cubic be replaced by a plane nodal cubic, the two double generators will lie in one plane, meeting in the node.

Let $wy^2 = x(x - w)^2$, $z = 0$ be $[c_3]$,

$$\begin{aligned} y_1 &= mx_1, \\ w_1 y_1^2 &= x_1 (x_1 - w_1)^2, \end{aligned} \qquad \frac{a'\mu^2 + b'\mu + b}{a\mu^2 + b\mu}.$$

The equation is

$$2\left[(ay - a'x)\left((w - x)\, 2zx - z^2 y\right) + (by - b'x)\, c_3\right]\left[-b'xz^2 - 2zx(x - w)\right] - \left((ay - a'x)\, z + c'c_3\right)^2 = 0.$$

The residual nodal curve is another $[c_3]$ having its node at $(0, 0, 0, 1)$. The symbol is $\overline{2c_1^3} + 2[c_3]^2 + 2g^2$.

The dual of all the unicursal S_6's with a tacnodal directrix are of one kind. The nodal curve consists of a fourfold directrix and two generators, the former counting as 8. Any plane section has two distinct tacnodes on it.

§3.—*Triple Directrix Line.*

(a). *No Double Generator.*

16. Let c_1 cut c_5 in a double point, and be in (1, 1) correspondence with it. c_1 is now simple directrix and double generator. The residual is a c_7^3, having two triple points, and cutting c_1 four times. [This was type XXXV.] No sub-forms can exist, since no double tangents can be drawn. Scrolls of this type are necessarily of the [3, 3] type, since c_5 must have a double point.

If c_1 and c_5 be in (2, 1) correspondence, the same type as before results except that now c_1 is double directrix and simple generator (old type XLV).

If c_1 and c_5 be in (3, 1) correspondence, such that both values of λ at the node correspond to the μ of that point, c_1 is a triple directrix (old type XXIII). This last type can be generated by developables of the form

$$\begin{aligned} at^3 + bt^2 + ct + d &= 0, \\ xt^2 + y &= 0. \end{aligned}$$

Using the same notation as on p. 73, Vol. XXV,

$$\begin{aligned} \psi_1 &= x\,\{(dx - ay)\,c + b^2y\} = 0, \\ \psi_2 &= \quad (dx - ay)^2 + bcxy = 0, \\ \psi_3 &= y\,\{(dx - ay)\,b - c^2x\} = 0. \end{aligned}$$

If, in particular, b, c pass through $(0, 0, 0, 1)$, one of the triple points on c_7 is on c_1 at this point.

If $a = a_1x + a_2y + a_3z + a_4w$ and similarly for the other terms, $c_4 = 0$, $b_4 = 0$, and it is no restriction if $c_3 : c_1 = b_2 : b_1$. If

$$a_4b_1 + b_2d_4 = 0,$$

the other triple point on c_7 becomes consecutive to the first one. All the branches have a common tangent at this point.

(b). *One Double Generator.*

17. If the two values of λ at a second node on c_5 correspond to the same value of μ in the (1, 2) correspondence, a double generator exists (type XLVI). The residual sextic cuts c_1 in three points. In particular, the c_6 may be replaced by a triple conic. If

$$\lambda = \frac{y}{x}, \quad \mu = \frac{yz}{yw - x^2}$$

and λ, μ be connected by a (2, 3) correspondence having 0, 0 for a double element, the resulting S_6 will have the symbol $(c_1^2 + g') + g^2 + c_2^3$.

The same two forms will exist if the correspondence be between a triple line and a plane c_5 or a triple c_2. In case a multiple point of c_6 lies on c_1, the g^2 may pass through this point, or the other one. The discussion is fully given by Bergstedt. If $b \equiv c$, the $c_6^3 \equiv c_3^3$.

(c). *Two Double Generators.*

18. If, in the (1, 2) correspondence between c_1, c_5, two branch-points exist at nodes, the symbol becomes $(c_1^2 + g') + 2g^2 + c_5^2$ (XLVII). Similarly for the other case. The two g^2 cannot intersect.

(d). *Three Double Generators.*

19. The line c_1 must now be a triple directrix, since a (1, 2) correspondence can only have two branch-points. The residual is a c_4 of the second kind. Finally, two of these may become tacnodal or the three may become oscnodal or all may unite in a triple generator. The last form requires that all three values of λ at the triple point of c_5 correspond to the same value of μ on c_1. It may also be shown as follows: Since any plane section of S_6 containing g^3 is a rational cubic, let

$$\begin{aligned}
&x = 1 - \lambda^2, \quad y = \lambda(1 - \lambda^2), \quad z = 0, \quad w = 1, \\
&x = a, \qquad\quad\; y = b, \qquad\qquad\quad\; z = \mu, \quad w = 1, \\
&\mu f_3(\lambda) + 1 - \lambda^2 - a + m\lambda(1 - \lambda^2) - mb = 0.
\end{aligned}$$

The S_6 is of the form required, since the three values of λ corresponding to $\mu = 0$ are collinear.

(e). *Four Double Generators.*

20. As S_6 of this form must belong to a linear congruence, only two types will be considered, as particular cases which were omitted before.

Given a c_5 having two distinct nodes P_1, P_2 and four consecutive double points at P. Draw a line c_1 through P_2 not in plane of c_5. Make the pencil whose vertex is P_1 projective with range on l in such a way that P_1P_2 corresponds to P_2. Connect each point of l with the three points in which the corresponding line through P_1 cuts c_5. The line l will be a triple line on the sextic scroll; there will be four consecutive double generators at P, hence the residual nodal line will be another triple directrix. An arbitrary plane through the singular generator will cut a quartic curve having an oscnode on the singular line. If, instead of four consecutive nodes at P, a tacnode and a simple branch passing through it be chosen, the singular generator will count as triple and consecutive nodal. Any plane passed through the line will have a nodal cubic as residual section. Similarly, scrolls of higher order can have all their double generators consecutive or coincident. The elimination is rational in each case.

§4.—*Fourfold Directrix Line.*

21. The enumeration given in my previous list is complete. They may all be generated by c_1 and c_6, the former passing through a triple point on the latter. When the point is a cusp, the residual quartic curve has a double point on c_1, and similarly for the cubic when a double generator exists. Sixteen forms exist. The correspondence may be (1, 1) without any corresponding element, or (2, 1) with a simple self-corresponding element, or (3, 1) with a double (really two simple elements, defining a branch-point) element, or, finally, (4, 1) with all three self-corresponding elements. In all except the first form one or two double generators may exist.

22. The dual of a tacnodal directrix is a fourfold line, such that the two generators which lie in any plane through the line intersect on the line.

Let $x_1, y_1, 0, w_1$ be a point on the c_5,

$$x^2yw^2 + x\mu_3(x, y)w + \mu_5 = 0, \quad z = 0,$$

which has a tacnodal point and a simple branch at the point (0, 0, 0, 1). Connect the point to the point $(0, 0, \mu, 1)$ on c_1,

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\mu w - z}{\mu w_1}$$

and let

$$y_1 = mx_1, \quad \mu = \frac{m}{am^2 + bm + c}.$$

The equation of the scroll is

$$(ax^2z + bxy(z - w) + cy^2z)^2 \\ + (ax^2z + bxy(z - w) + cy^2z)u_3(x, y) + yu_5(x, y) = 0.$$

The line x, y is a double tacnodal directrix through each point of which pass four generators. Any plane section cuts this line in a pair of distinct tacnodal branches, hence it is equivalent to a nodal curve of order 8, when c_5 has no further singular points $p = 2$, and the surface has no other nodal line. If $u_3 = u_1v_2$ and $u_5 = u_1^2v_3$, one double generator exists, and if $u_3 = u_1v_1w_1$, $u_5 = u_1^2v_1^2t_1$, two such may exist. The former is of genus 1 and the latter of genus 0.

By replacing c_5 by a c_3 of the form

$$yw^2 + u_2w + u_3 = 0, \quad z = 0$$

and letting

$$y_1 = mx_1, \quad \mu = m^2,$$

a S_6 is obtained in which the directrix has the same form as before. The equation is

$$(wy^2 - x^2z)^2 + y(wy^2 - x^2z)u_2(x, y) + y^3u_3(x, y) = 0,$$

which is of genus 1. When $u_2 = u_1v_1$ and $u_3 = u_1^2w_1$, a double generator appears. The line $y = 0$, $z = 0$ is a cuspidal generator. If $a = 0$ and $b = 0$, the line $x = 0$, $y = 0$ is a double generator.

Four types of the unicursal S_6 exist. When the two double points on c_5 are distinct, the generators passing through them may be skew or intersect on the directrix, when the two planes form a pair in the involution. Similarly, if the two nodes form a tacnode, the generators do not lie in the same plane unless the plane be a double plane of the involution.

The second and fourth of these surfaces can, therefore, be generated by a (1, 4) correspondence between a conic and a straight line which it does not intersect.

If $\mu = \dfrac{\phi_3(m)}{\psi^3(m)}$, the most general form of the equation is

$$(\phi_2(x, y)\, w - z\psi_2(x, y))^3 = xy\, [\phi_3(x, y)]^2.$$

If ϕ_2 is a square, the double generators are tacnodal.

5. *Fivefold Directrix Line.*

23. The twelve forms mentioned complete the list. They can all be generated by c_1 and c_5 in $(k, 1)$ correspondence, $k = 1$ to 5.

B. *Scrolls of Genus One.*

24. The S_6 of genus 1, which were omitted, are similar to the corresponding unicursal scrolls having a double rectilinear directrix. If, on p. 80, $e = c = 0$, the common chord is a double generator. If $a = 1$, $f = bg$, a c_1^2, also appears. Its symbol is $c_1^2 + [c_3^2] + 2c_2^2 + g^2$.

This scroll can also be generated as follows:

Given the non-singular cubic c_3,

$$y^2 w = x^3 - xw^3, \quad z = 0$$

and the line c_1, $x = 0$, $y = 0$ passing through the intersection of c_3 and one of its harmonic polars. Join the point $(0, 0, z_2, w_2)$ on c_1 to $(x_1, y_1, 0, w_1)$ on c_3.

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{wz_2 - zw_2}{z_2},$$

from which $$y^2 x_1 = x^3 (x_1^2 - w_1^2), \quad xz_2 = x_1 z_2 w - x_1 z.$$

Let the parameters x_1, z_2 be connected by the relation

$$x_1 = z_2^2.$$

The resulting S_6 will have the equation

$$\begin{vmatrix} x^2 & 0 & -y^2 & 0 & -x & 0 \\ 0 & x^2 & 0 & -y^2 & z & x \\ w & -z & -x & 0 & 0 & 0 \\ 0 & w & -z & -x & 0 & 0 \\ 0 & 0 & w & -z & -1 & 0 \\ 0 & 0 & 0 & w & 0 & 1 \end{vmatrix} = 0.$$

c_1 and c_3 will each be double curves. There is one g^2, $x = 0$, $w = 0$. In the planes $z \pm iw = 0$ lie the c_2^2, $2x^2 \pm i(y^2 + z^2) = 0$, hence the symbol is $c_1^2 + [c_3^2] + g^2 + 2c_2^2$. Since the section made by the plane $w = 0$ is g^2 and the tacnodal quartic

$$z^4 - x^4 + z^2y^2 = 0,$$

hence S_6 is of genus 1, and has no further nodal curve.

Similarly, if a c_1 and an elliptic c_5 be put in (2, 1) correspondence with a tacnodal generator, the symbol becomes $c_1^2 + \overline{2g}^2 + c_6^2$. The c_1 may lie on a double tangent, or on two or three. In the latter case c_6 is replaced by a c_4 and a c_2, the latter being tangent to the singular generator. The c_4 is of the first kind and cuts c_2 in two points not on $\overline{2g}^2$.

My form VII on p. 96 is incorrect. This S_6 is unicursal and contains a g^3. The forms with a tacnodal directrix are most easily derived from the dual.

C. *Scrolls of Genus Greater than One.*

25. The only form omitted in $p = 2$ is $c_1^2 + c_4^2 + [c_3^2]$. The cubic is non-singular and cuts c_1. c_4 is of genus 1 and cuts c_3 in four points. The surface can be generated by an elliptic (2, 2) correspondence between c_1 and c_3, the point of intersection being a double self-corresponding point. This form may be generated as follows:

Given a non-singular cubic curve and a point P upon it, not on a harmonic polar. Let Q be the first tangential of P. Let a range on a straight line passing through P, but not lying in the plane of the cubic, be made projective with the pencil whose vertex is Q in the plane of the cubic. The line PQ of the pencil should correspond to the point P of the range. Finally, with each point of the range should be associated the point symmetric to it with regard to P. Lines joining the points of this double range to each of the points in which the corresponding line of the pencil cut the cubic will be a S_6 having the symbol $c_1^2 + [c_3^2] + c_4^2$ and of genus 2. E. g. given $y^2w = x^3 - 12xw^2$, $z = 0$ and $p \equiv (2, 4, 0, 1)$. Then $Q \equiv (4, 4, 0, 1)$. The correspondence is expressed by

$$\begin{aligned} x &= -2w, & x - 4w &= \lambda(x - 4w), \\ y &= 4w, & \lambda &= \mu^2, \\ z &= \mu w. \end{aligned}$$

By writing

$$y^2w - x^3 + 12x = c_3, \quad 24zw^2 - 24zxw - 6xz - 8yzw - zy^2 = a,$$
$$4z^2\{2(y-4) - 3x\} = b,$$

the equation of S_6 becomes, multiplied by $[y - 4w + az]^2$,

$$[(4w - y)c + b(4w - x)]^2 = [a(4w - y) - 6bz] . [a(x - 4w) - 6cz].$$

The quartic curve is of the first kind, and does not cut c_1.

Similarly, if P be a point of inflexion, and Q coincide with P, the four generators which lie in the same plane through c_1 will intersect in pairs on c_1. Let

$$y_1w_1^2 = x_1^3 - x_1y_1w_1, \quad z = 0, \text{ be } c_3,$$
$$y_1 = \mu^2 x_1 \text{ be the pencil } Q,$$
$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\mu - z}{\mu} \text{ be the line joining corresponding points.}$$

The equation of the surface is

$$x\sqrt{x - y} = \sqrt{y}w - \sqrt{x}z,$$

or, rationalized,

$$[x^2(x - y) - yw^2 - xz^2]^2 = 4w^2z^2xy.$$

The section of the surface made by $w = 0$ is a cubic of exactly the same form as the given one. The directrix $x = 0$, $y = 0$ is not a generator, but every plane

section will have a tacnode at the point of intersection with it, and three other nodes on each cubic. The surface is of genus 2 and symbol $c^2_{1,3} + 2\,[c^2_3]$.

26. The general dual of the form treated in §13 may be generated as follows: To the range $(0, 0, \mu, 1)$ corresponds the axial pencil $\mu z + w = 0$, and to the points of c_5 correspond the planes

$$x_1 x + y_1 y + w = 0\,,$$

wherein
$$x_1^3 y_1 w_1^2 + x_1 u_3 (x_1,\, y_1)\, w_1 + u_5 (x_1,\, y_1) = 0\,,$$

$$y_1 = m x_1\,, \quad \mu = \frac{m}{am^2 + bm + c}\,.$$

From these equations we obtain

$$\begin{aligned} &awm^2 + (bw + z)\, m + cw = 0\,,\\ &m\,(x + my)^3 - wu_3\,(m)(x + my) + w^2 u_5\,(m) = 0\,. \end{aligned}$$

The m eliminant is the surface required, after dividing out the extraneous factor w^3. The form of this equation is sufficiently general to include all the scrolls having a tacnodal directrix, but it is easier to obtain the dual of each particular case directly. Of those of genus 1 and simple residual nodal curve, two forms exist: one double generator and c^2_6 or a tacnodal double generator and a c^2_5. The other forms have already been derived directly.

Table of Forms.

27. In the following list of types, group A includes all the unicursal ones, while groups B, C contain only those which were not included in the previous lists. The notation for the symbol of the type is the same as that employed before, the number in the next column refers to the paragraph of the present paper in which the corresponding type is derived. Finally, the numbers 2, 3 in the last column of A give the class of the simplest developable which all the generators of the scroll touch. Those having the number 2 are of [2, 4] type, while those marked 3 are of [3, 3] type. The enumeration is now believed to be complete.

A.

1, 2, 3	$c_1^1 + c_{10,6}^2$	2	2
4, 5, 6	$c_1^1 + c_{10,3}^2$	3	3
7	$c_1^1 + 2c_{5,3}^2$	2	2
8	$c_1^1 + 2c_{5,3}^2$	3	3
9, 10	$c_1^2 + c_{9,6}^2$	4	2
11	$c_1^2 + c_{6,4}^2 + [c_{3,2}^2]$	4	2
12	$c_1^2 + 3\,[c_{3,2}^2]$	4	2
13, 14	$c_1^2 + c_{9,3}^2$	5	3
15	$c_1^2 + c_{6,3}^2 + c_3^2$	5	3
16	$c_1^2 + 3c_3^2$	5	3
17–24	replace c_1^2 by $(c_1^1 + g^1)$ in 9–16	5	
25, 26	$c_1^2 + c_{8,5}^2 + g^2$	6	2
27	$c_1^2 + c_{6,4}^2 + c_2^2 + g^2$	6	2
28	$c_1^2 + c_{5,3}^2 + [c_3^2] + g^2$	6	2
29	$c_1^2 + c_2^2 + 2\,[c_3^2] + g^2$	6	2
30 31	$c_1^2 + c_{8,3}^2 + g^2$	7	3
32	$c_1^2 + c_{6,3}^2 + c_2^2 + g^2$	7	3
33	$c_1^2 + c_{5,2}^2 + c_3^2 + g^2$	7	3
34	$c_1^2 + c_2^2 + 2c_3^2 + g^2$	7	3
35, 36	$c_1^2 + c_{7,4}^2 + 2g^2$	8	2
37, 38	$c_1^2 + c_{7,4}^2 + \overline{2g^2}$	8	2
39, 40	$c_1^2 + c_{7,3}^2 + 2g^2$	9	3
41, 42	$c_1^2 + c_{7,3}^2 + \overline{2g^2}$	9	3
43	$c_1^2 + c_{4,2}^2 + [c_{3,2}^2] + 2g^2$	9	2
44	$c_1^2 + c_{4,2}^2 + [c_{3,2}^2] + \overline{2g^2}$	9	2
45	$c_1^2 + c_{5,3}^2 + c_2^2 + \overline{2g^2}$	9	2
46	$c_1^2 + [c_{3,2}^2] + 2c_2^2 + \overline{2g^2}$	9	2
47	$c_1^2 + c_4^2 + c_3^2 + 2g^2$	10	3
48	$c_1^2 + c_{5,2}^2 + c_2^2 + 2g^2$	10	3
49	$c_1^2 + c_3^2 + 2c_2^2 + 2g^2$	10	3
50	$c_1^2 + c_{5,2}^2 + c_2^2 + \overline{2g^2}$	10	3
51	$c_1^2 + c_3^2 + 2c_2^2 + \overline{2g^2}$	10	3

52	$c_1^3 + c_{6,3}^2 + \overline{3g^3}$	11	2
53	$c_1^2 + c_{6,3}^3 + g^2 + \overline{2g^2}$	11	3
54	$c_1^2 + c_4^2 + c_2^3 + g^3 + \overline{2g^2}$	11	3
55	$c_1^2 + c_{4,2}^2 + c_3^2 + \overline{3g^2}$	11	2
56	$c_1^3 + 3c_2^2 + g^3 + \overline{2g^3}$	11	3
57	$c_1^3 + c_4^2 + \overline{3g^3}$	12	2
58	$\overline{2c_1^2} + c_5^3 + \overline{3g^2}$	13	2
59	$\overline{2c_1^2} + c_6^2 + 2g^3$	13	2
60	$\overline{2c_1^2} + c_6^2 + 2g^3$	13	3
61	$\overline{2c_1^3} + c_5^3 + g^3 + \overline{2g^3}$	13	3
62	$\overline{2c_1^2} + c_4^3 + 2\,.\,\overline{2g^3}$	13	3
63	$\overline{2c_1^3} + 2c_3^3 + 2\,.\,2g^3$	13	3
64	$\overline{2c_1^3} + c_2^2 + [c_{3,2}^2] + \overline{2g^2} + g^2$	14	3
65	$\overline{2c_1^3} + c_3^2 + [c_{3,2}^3] + \overline{3g^3}$	14	2
66	$\overline{2c_1^2} + 2c_3^2 + 2g^2$	15	3
67	$\overline{2c_1^3} + 2\,[c_{3,2}^2] + 2g^3$	15	2
68	$(c_1^1 + g^2) + c_{7,3}^2$	16	3
69	$(c_1^2 + g^1) + c_{7,8}^3$	16	3
70	$c_1^3 + c_{7,3}^2$	16	3
71	$(c_1^2 + g^1) + c_{6,2}^2 + g^2$	17	3
72	$(c_1^3 + g^1) + c_2^3 + g^3$	17	3
73	$c_1^3 + c_{6,2}^2 + g^3$	17	3
74	$c_1^3 + c_2^3 + g^3$	17	3
75	$(c_1^2 + g^1) + c_{5,2}^3 + 2g^3$	18	3
76	$c_1^3 + c_{5,2}^2 + 2g^3$	18	3
77	$c_1^3 + c_4^3 + 3g^3$	19	3
78	$c_1^3 + c_4^2 + \overline{2g^3} + g^2$	19	3
79	$c_1^3 + c_4^2 + \overline{3g^3}$	19	3
80	$c_1^3 + c_4^3 + g^3$	19	3
81	$c_1^3 + c_1^4 + \overline{4g^3}$	20	2
82–97	$c_1^4 +$ .... given before.	21	2
98, 99	$\overline{2c_1^4} + 2g^3$	22	2
100, 101	$\overline{2c_1^4} + \overline{2g^3}$	22	2

102–113	Fivefold line. Vol. XXV, p. 83	23	1
114–122	Obvious forms contained in a linear congruence, not previously mentioned in this paper.		

B.

1	$c_1^3 + [c_8^2] + 2c_3^2 + g^3$	24	
2, 3	$c_1^3 + c_6^2 + \overline{2g^3}$	24	
4	$c_1^3 + c_3^2 + c_4^3 + \overline{2g^3}$	24	
5, 6	$\overline{2c_1^3} + c_{6,2}^2 + g^3$	26	
7, 8	$\overline{2c_1^3} + c_5^3 + 2g^2$	26	
9	$\overline{2c_1^2} + 2c_3^3 + c_4^2 + g^2$	14	
10	$\overline{2c_1^2} + [c_8^2] + c_3^2 + \overline{2g^3}$	14	
11	$2c_1^4 + g^3$	22	37 forms.

C.

1	$c_1^2 + c_4^2 + [c_8^2]$	25	
2	$\overline{2c_1^3} + c_6^2$	26	
3	$\overline{2c_1^3} + 2\,[c_8^2]$	25	
4	$\overline{2c_1^4}$	26	14 forms.
			6 forms, $p > 2$.

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On a Class of Differential Equations.

By Alexander Chessin.

---

1. Let us denote by $D^{(n)}$ the operation

$$A_0 \frac{d^n}{dx^n} + A_1 \frac{d^{n-1}}{dx^{n-1}} + \ldots\ldots + A_{n-1} \frac{d}{dx} + A_n,$$

where the coefficients $A_0, A_1, \ldots\ldots, A_n$ are functions of x, and consider a function y of x defined by the linear differential equation of order mn,

$$y_m + a_1 y_{m-1} + a_2 y_{m-2} + \ldots\ldots + a_m y = f(x), \tag{1}$$

with the constant coefficients $a_1, a_2, \ldots\ldots, a_m$, and where

$$y_k = D^{(n)} y_{k-1}; \quad y_0 = y; \qquad k = 1, 2, \ldots\ldots, m. \tag{2}$$

The integration of (1) can be reduced to the integration of an equation of the form

$$D^{(n)} y = f(x), \tag{3}$$

2. To show this, let us make the substitution

$$y_1 = -\lambda y + z, \tag{4}$$

from which it follows that

$$y_k = -\lambda y_{k-1} + z_{k-1}; \qquad k = 1, 2, \ldots\ldots, m; \tag{4 bis}$$

the functions $z_1, z_2, \ldots\ldots$ being defined by equations similar to (2), viz.

$$z_k = D^{(n)} z_{k-1}; \quad z_0 = z; \qquad k = 1, 2, \ldots\ldots, m-1.$$

Then equation (1) becomes

$$\begin{aligned} z_{m-1} + (a_1 - \lambda) z_{m-2} + (a_2 - a_1\lambda + \lambda^2) z_{m-3} + \dots \\ + [a_{m-1} - a_{m-2}\lambda + a_{m-3}\lambda^2 - \dots + (-1)^{m-1}\lambda^{m-1}] z \\ + [a_m - a_{m-1}\lambda + a_{m-2}\lambda^2 - \dots + (-1)^m\lambda^m y] = f(x), \end{aligned}$$

and, by giving to λ the value of any one of the roots of the equation

$$a_m - a_{m-1}\lambda + a_{m-2}\lambda^2 - \dots + (-1)^m\lambda^m = 0, \tag{5}$$

we arrive at an equation similar to (1) but of order $(m-1)n$, viz.

$$z_{m-1} + b_1 z_{m-2} + b_2 z_{m-3} + \dots + b_{m-1} z = f(x), \tag{6}$$

where

$$b_k = a_k - a_{k-1}\lambda + a_{k-2}\lambda^2 - \dots + (-1)^k\lambda^k. \tag{7}$$

3. Now, suppose $[z]$ to be the general solution of (6). It will contain $(m-1)n$ arbitrary constants. To obtain the function y, we only need to substitute $[z]$ for z into (4) and integrate the resulting equation

$$y_1 = -\lambda y + [z] \tag{8}$$

which is, obviously, of the form (3). This integration will introduce n new arbitrary constants, bringing the total number of them to mn. The integral so obtained is, therefore, the general solution of the given differential equation.

On the other hand, the general solution of (6) may be obtained by a series of successive and similar operations. Thus, the substitution

$$z_1 = -\mu z + u, \tag{9}$$

with the condition that μ be a root of the equation

$$b_{m-1} - b_{m-2}\mu + b_{m-3}\mu^2 - \dots + (-1)^{m-1}\mu^{m-1} = 0 \tag{10}$$

reduces the integration of (6) to the integration of the differential equation of order $(m-2)n$,

$$u_{m-2} + c_1 u_{m-3} + c_2 u_{m-4} + \dots + c_{m-2} u = f(x), \tag{11}$$

where

$$c_k = b_k - b_{k-1}\mu + b_{k-2}\mu^2 - \dots + (-1)^k\mu^k, \tag{12}$$

and $[u]$ being the general solution of (11), the general solution of (6) is obtained by the integration of the equation

$$z_1 = -\mu z + [u]$$

which, again, is of the form (3).

By means of these successive reductions, we will finally arrive at an equation of the form

$$w_1 + \theta w = f(x), \tag{13}$$

i. e. of the form (3). Q. E. D.

4. In practice, however, it would not be convenient to proceed in this manner. We will show how the single integration of the equation (13) at once furnishes the general solution of (1).

To begin with, it can be shown that equation (13) holds good for all the values of θ which satisfy the equation $F(\theta) = 0$, where

$$F(\theta) \equiv a_m - a_{m-1}\theta + a_{m-2}\theta^2 - \ldots + (-1)^m\theta^m.$$

In fact, the values of θ must satisfy the equation resulting from the elimination of $\lambda, \mu, \nu, \xi, \ldots$ between the equations (5), (10) and those formed successively in a similar manner, viz.

$$\left.\begin{array}{l} c_{m-2} - c_{m-3}\nu + c_{m-4}\nu^2 - \ldots + (-1)^{m-2}\nu^{m-2} = 0, \\ d_{m-3} - d_{m-4}\xi + d_{m-5}\xi^2 - \ldots + (-1)^{m-3}\xi^{m-3} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{14}$$

where $d_k = c_k - c_{k-1}\nu + c_{k-2}\nu^2 - \ldots + (-1)^k\nu^k$, etc. Now, the form of the coefficients in these equations enables us to effect the elimination successively in a very simple manner. In general, let

$$\begin{aligned} \Phi(\eta) &= \alpha_p - \alpha_{p-1}\eta + \alpha_{p-2}\eta^2 - \ldots + (-1)^p\eta^p, \\ \Psi(\eta, \zeta) &= \beta_{p-1} - \beta_{p-2}\zeta + \beta_{p-3}\zeta^2 - \ldots + (-1)^{p-1}\zeta^{p-1}, \end{aligned}$$

where $\beta_k = \alpha_k - \alpha_{k-1}\eta + \alpha_{k-2}\eta^2 - \ldots + (-1)^k\eta^k$. Then

$$\Phi(\eta) = (\zeta - \eta)\Psi(\eta, \zeta) + \Phi(\zeta),$$

and it is clear that the result of the elimination of η between the equations

$$\Phi(\eta) = 0, \quad \Psi(\eta, \zeta) = 0;$$

is the equation
$$\Phi(\zeta) = 0.$$

Applying this remark to the successive elimination of .... ξ, ν, μ, λ between the equations (14), (10), (6), we readily see that the result of the elimination is the equation

$$F(\theta) = 0;$$

in other words, θ can be any one of the roots of the equation (6).

5. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of (5), and let us, first, assume that no two of them are equal. To each value λ_k of θ corresponds a distinct integral $[w]_k$ of (13), containing n arbitrary constants. It can be shown that in this case the general solution of (1) is given by the formulas

$$y = \sum_{k=1}^{k=m} C_k [w]_k; \quad C_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h_{k, m-1}};$$

$$\Delta = \begin{vmatrix} h_{1,0} & h_{1,1} & \ldots & h_{1,m-1} \\ h_{2,0} & h_{2,1} & \ldots & h_{2,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ h_{m,0} & h_{m,1} & \ldots & h_{m,m-1} \end{vmatrix}; \tag{15}$$

$$h_{k,e} = \lambda_k^e; \qquad k = 1, 2, \ldots, m; \; e = 0, 1, \ldots, m-1.$$

6. To show this, we recall the well-known relations in the theory of determinants

$$\sum_{k=1}^{k=m} C_k h_{k,e} = 0, \qquad e = 0, 1, \ldots, m-2;$$

$$\sum_{k=1}^{k=m} C_k h_{k,m-1} = (-1)^{m-1};$$

or, introducing the values of $h_{k,e}$,

$$\left.\begin{aligned}&\sum_{k=1}^{k=m} C_k\lambda_k^e = 0, \qquad e = 0, 1, \ldots, m-2;\\&\sum_{k=1}^{k=m} C_k\lambda_k^{m-1} = (-1)^{m-1}.\end{aligned}\right\} \tag{16}$$

The operation $D^{(n)}$ performed on the expression (15) of y gives

$$y_1 = \sum_{k=1}^{k=m} C_k\,[w_1]_k,$$

and, by (13), therefore,

$$y_1 = -\sum_{k=1}^{k=m} C_k\lambda_k\,[w]_k + f(x)\sum_{k=1}^{k=m} C_k.$$

But, by the first of the relations (16), viz. for $e = 0$, we have $\Sigma C_k = 0$, hence

$$y_1 = -\sum_{k=1}^{k=m} C_k\lambda_k\,[w]_k. \tag{17}$$

Likewise, the operation $D^{(n)}$ performed on (17) gives, by (13) and the relation (16) for $e = 1$,

$$y_2 = \sum_{k=1}^{k=m} C_k\lambda_k^2\,[w]_k.$$

Continuing the application of the symbol $D^{(n)}$ successively we find that

$$\begin{aligned}&y_e = (-1)^e\sum_{k=1}^{k=m} C_k\lambda_k^e\,[w]_k, \qquad e = 0, 1, \ldots, m-1;\\&y_m = (-1)^m\sum_{k=1}^{k=m} C_k\lambda_k^m\,[w]_k + f(x),\end{aligned}$$

from which it follows that

$$\begin{aligned}&y_m + a_1y_{m-1} + a_2y_{m-2} + \ldots + a_my\\&\qquad = \sum_{k=1}^{k=m} C_k\,[w]_k\,\{a_m - a_{m-1}\lambda_k + a_{m-2}\lambda_k^2 - \ldots + (-1)^m\lambda_k^m\} + f(x),\end{aligned}$$

and, λ_k being a root of (5), the right-hand side of the last equation reduces to $f(x)$. Q. E. D.

7. We pass now to the case of equal roots. Let us, for the present, assume that $\lambda_1=\lambda_2=\ldots\ldots=\lambda_i$, and that all the other roots of (5) are distinct. In this case the general solution of (1) is given by the formulas

$$y=\sum_{k=1}^{k=i} C_k \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}}+\sum_{k=1}^{k=m-i} C'_{i+k}[w]_{i+k}; \tag{18}$$

where the arbitrary constants in $[w]_1$, $\frac{d[w]_1}{d\lambda_1}$, $\frac{d^2[w]_1}{d\lambda_1^2}$, ..., $\frac{d^{i-1}[w]_1}{d\lambda_1^{i-1}}$ are *independent of one another*. Thus, if

$$[w]_1=\phi(x, \lambda_1, a_1, a_2, \ldots\ldots, a_n),$$

we will understand under $\frac{d[w]_1}{d\lambda_1}$ the function

$$\frac{d\phi(x, \lambda_1, a'_1, a'_2, \ldots\ldots, a'_n)}{d\lambda_1},$$

with arbitrary constants a'_1, a'_2, ..., a'_n independent of a_1, a_2, ..., a_n.

As to the coefficients C_k and C'_{i+k}, they will be determined by the formulas

$$C_k=(-1)^{m-1}\frac{\partial \log\Delta}{\partial h_{k,m-1}}, \quad C'_{i+k}=(-1)^{m-1}\frac{\partial \log\Delta}{\partial h_{i+k,m-1}};$$

$$\Delta=\begin{vmatrix} h_{1,0} & h_{1,1} & \ldots\ldots & h_{1,m-1} \\ h_{2,0} & h_{2,1} & \ldots\ldots & h_{2,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ h_{m,0} & h_{m,1} & \ldots\ldots & h_{m,m-1} \end{vmatrix}; \tag{19}$$

$$h_{k,e}=\frac{d^{k-1}\lambda_1^e}{d\lambda_1^{k-1}}, \qquad k=1, 2, \ldots\ldots, i;$$

$$h_{i+k,e}=\lambda_{i+k}^e, \qquad k=1, 2, \ldots\ldots, m-i, \qquad e=0, 1, 2, \ldots\ldots, m-1.$$

Moreover, we will assume that $C_k=0$ for $k>i$, and $C'_j=0$ for $j<i+1$.

8. To prove that (18) is the general solution of (1) in this case, we observe again that

$$\left.\begin{aligned}&\sum_{k=1}^{k=i} C_k h_{k,e} + \sum_{k=1}^{k=m-i} C'_{i+k} h_{i+k,e} = 0, \qquad e = 0, 1, \ldots, m-2;\\&\sum_{k=1}^{k=i} C_k h_{k,m-1} + \sum_{k=1}^{k=m-i} C'_{i+k} h_{i+k,m-1} = (-1)^{m-1}.\end{aligned}\right\} \qquad (20)$$

We shall, besides, make use of the following relation which is based on the principle of interchangeability of the order of differentiation and on the equation (13):

$$D^{(n)} \frac{d^k[w]_1}{d\lambda_1^k} = \frac{d^k D^{(n)}[w]_1}{d\lambda_1^k} = \frac{d^k[w_1]_1}{d\lambda_1^k} = -\lambda_1 \frac{d^k[w]_1}{d\lambda_1^k} - k\frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}}. \qquad (21)$$

9. The operation $D^{(n)}$ performed on the expression (18) of y gives, account being taken of the relation (21),

$$y_1 = -\sum_{k=1}^{k=i}(kC_{k+1} + \lambda_1 C_k)\frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} - \sum_{k=1}^{k=m-i} C'_{i+k}\lambda_{i+k}[w]_{i+k}$$
$$+ f(x)\left\{C_1 + \sum_{k=1}^{k=m-i} C'_{i+k}\right\}.$$

But by the first of the relations (20), viz. for $e = 0$, $C_1 + \Sigma C'_{i+k} = 0$; hence

$$y_1 = -\sum_{k=1}^{k=i}(kC_{k+1} + \lambda_1 C_k)\frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} - \sum_{k=1}^{k=m-i} C'_{i+k}\lambda_{i+k}[w]_{i+k}. \qquad (22)$$

Continuing the application of the symbol $D^{(n)}$ successively to $y_1, y_2, \ldots,$ we find, in general,

$$y_e = (-1)^e \sum_{k=1}^{k=i} M_{k,e} \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + (-1)^e \sum_{k=1}^{k=m-i} C'_{i+k}\lambda_{i+k}^e [w]_{i+k}, \qquad (23)$$

where

$$(k-1)!\,M_{k,e}=(k+e-1)!\,C_{k+e}+\binom{e}{1}(k+e-2)!\,C_{k+e-1}\lambda_1+\dots$$
$$+\binom{e}{j}(k+e-j-1)!\,C_{k+e-j}\lambda_1^j+\dots$$
$$+\binom{e}{1}k!\,C_{k+1}\lambda_1^{e-1}+(k-1)!\,C_k\lambda_1^e; \quad (24)$$
$$e=0,\,1,\,2,\,\dots,\,m-1;$$

and

$$y_m=(-1)^m\sum_{k=1}^{k=m}M_{k,m}\frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}}+(-1)^m\sum_{k=1}^{k=m-i}C'_{i+k}\lambda_{i+k}^m[w]_{i+k}+f(x). \quad (25)$$

10. From the preceding formulas it follows that

$$\begin{aligned}y_m+a_1y_{m-1}+a_2y_{m-2}+\dots+a_my&=f(x)\\&+(-1)^m\sum_{k=1}^{k=i}\frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}}\{M_{k,m}-a_1M_{k,m-1}+a_2M_{k,m-2}-\dots+(-1)^m a_mM_{k,0}\}\\&+(-1)^m\sum_{k=1}^{k=m-i}C'_{i+k}[w]_{i+k}\{\lambda_{i+k}^m-a_1\lambda_{i+k}^{m-1}+a_2\lambda_{i+k}^{m-2}-\dots+(-1)^ma_m\}.\end{aligned}$$

But, λ_{i+k} being a root of (5), the last sum on the right-hand side of this equation vanishes, and to prove that (18) is the general solution of (1), we, therefore, only need to show that

$$M_{k,m}-a_1M_{k,m-1}+a_2M_{k,m-2}-\dots+(-1)^mM_{k,0}=0. \quad (26)$$

Now, it is easy to verify that the left-hand side of (26) may be presented in the form

$$\sum_{e=0}^{e=m}\frac{(k+e-1)!}{(k-1)!\,e!}C_{k+e}\frac{d^eF(\lambda_1)}{d\lambda_1^e}.$$

But, λ_1 being a multiple root of (5) of the order i, we have

$$\frac{d^eF(\lambda_1)}{d\lambda_1^e}=0,\ \text{for } e=0,\,1,\,2,\,\dots,\,i-1;$$

on the other hand, for $e \geqq i$, since k is at least equal to one, $C_{k+e} = 0$ (see 7). Therefore, the left-hand side of (26) vanishes for all the values of $k = 1, 2, \ldots, i$. Q. E. D.

11. It can be readily seen now that, in the most general case, when (5) admits of several multiple roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ of respective orders $\alpha_1, \alpha_2, \alpha_3, \ldots$, with the condition

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots = m,$$

the general solution of (1) will be given by the formulas

$$y = \sum_{k=1}^{k=\alpha_1} C'_k \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + \sum_{k=1}^{k=\alpha_2} C''_k \frac{d^{k-1}[w]_2}{d\lambda_2^{k-1}} + \sum_{k=1}^{k=\alpha_3} C'''_k \frac{d^{k-1}[w]_3}{d\lambda_3^{k-1}} + \ldots; \tag{27}$$

$$C'_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h'_{k,m-1}}, \quad k = 1, 2, \ldots, \alpha_1; \quad h'_{k,e} = \frac{d^{k-1}\lambda_1^e}{d\lambda_1^{k-1}};$$

$$C''_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h''_{k,m-1}}, \quad k = 1, 2, \ldots, \alpha_2; \quad h''_{k,e} = \frac{d^{k-1}\lambda_2^e}{d\lambda_2^{k-1}};$$

$$C'''_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h'''_{k,m-1}}, \quad k = 1, 2, \ldots, \alpha_3; \quad h'''_{k,e} = \frac{d^{k-1}\lambda_3^e}{d\lambda_3^{k-1}};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta = \begin{vmatrix} h'_{1,0} & h'_{1,1} & \ldots & h'_{1,m-1} \\ h'_{2,0} & h'_{2,1} & \ldots & h'_{2,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h'_{\alpha_1,0} & h'_{\alpha_1,1} & \ldots & h'_{\alpha_1,m-1} \\ h''_{1,0} & h''_{1,1} & \ldots & h''_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h''_{\alpha_2,0} & h''_{\alpha_2,1} & \ldots & h''_{\alpha_2,m-1} \\ h'''_{1,0} & h'''_{1,1} & \ldots & h'''_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h'''_{\alpha_3,0} & h'''_{\alpha_3,1} & \ldots & h'''_{\alpha_3,m-1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

12. An important case of the equation (1), considered in this paper, occurs in some problems of elasticity, namely, when $n = 2$ and

$$D^{(2)} \equiv \frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{p^2}{x^2}.$$

The problem is then reduced to the integration of a Bessel equation with second member

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(\frac{p^2}{x^2}-\lambda\right)y=f(x).$$

This special case has been discussed by the author in a paper before the Academy of Sciences of Paris.*

* Sur une classe d'équations différentielles réductibles à l'équation de Bessel. C. R. 11 Mai, 1903. See also a paper by the author in C. R. 27 Octobre, 1902: Sur l'équation de Bessel avec second membre.

Surfaces with the Same Spherical Representation of their Lines of Curvature as Pseudospherical Surfaces.

By L. P. Eisenhart.

Introduction.

Surfaces whose Gaussian curvature is negative and constant—pseudospherical surfaces—have been the object of study of the geometers since the days of Gauss, and as the result of their study there exists to-day an extensively developed theory of the properties of these surfaces and the transformations by which groups of them can be obtained from one of them. The character of the lines of curvature on these surfaces is of interest, and especially the curves on the unit sphere which serve for their spherical representation. It can be shown that when these lines are parametric, the linear element of the sphere is reducible to the form

$$d\sigma^2 = \sin^2\omega\, du^2 + \cos^2\omega\, dv^2, \tag{a}$$

where ω is a particular solution of

$$\frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2} = \sin\omega\cos\omega. \tag{b}$$

In spite of the large amount of work which has been done upon these surfaces very little attention has been directed to the surfaces other than pseudospherical whose lines of curvature have this representation upon the sphere. It is our purpose to make a study of these surfaces and we shall find that many of the the theorems for pseudospherical surfaces have significant analogues in this

broader field. For convenience, we shall refer to them as the *A-surfaces*. After the manner of Codazzi, we write the linear element of the surface in the form

$$ds^2 = A^2du^2 + C^2dv^2. \tag{c}$$

In §1, several methods are discussed by means of which one can determine from a given A-surface all the surfaces with the same representation of their lines of curvature. It is found that the simplest solution of this problem reduces to the integration of a partial differential equation of the Laplace type. It will be convenient to refer to all the surfaces with the same spherical representation of their lines of curvature as forming a *group* or *class*.

In §2, we consider particular A-surfaces and notice first of all that the *moulure* surfaces are of this kind. When the functions A and C in (c) are given the respective values $\cos\omega$ and $\sin\omega$, the corresponding surface is pseudospherical. In seeking to determine all the A-surfaces for which A and C are functions of ω alone, we find that they are the pseudospherical surfaces and their parallels. The section closes with the determination of the A-surfaces, which are Weingarten surfaces.

The method of the *transformation of Lie* for pseudospherical surfaces can be applied at once to all the A-surfaces, with the result that when one of these surfaces is known, the spherical representation of an infinity of groups of new A-surfaces can be found directly; this is considered in §3.

Bianchi was the first to call attention to the fact that, if upon a surface of total curvature minus one, a family of parallel geodesics be given and upon tangents to these curves points be taken at unit distance from the point of contact, the locus of these points is a surface of the same total curvature and the lines are tangent to the second surface as well as to the first. The lines of curvature upon the new surface correspond to the similar lines on the original surface, and the linear element of the spherical representation of the former is given by (a) when ω is replaced by θ, the latter function being the angle between the tangent to the geodesic and the line of curvature $v = \text{const.}$ on the first surface. The determination of this angle θ requires the integration of a Riccati equation, so that there is an infinity of such families of geodesics. Darboux has given the name *transformations of Bianchi* to the foregoing. But one of these new surfaces can be defined also as the envelope of the plane perpendicular to the tangent plane

to the surface, through the point of contact and meeting the tangent plane in the line which makes an angle θ with the tangent to the line of curvature, v=const. It is found that the Riccati equation for the determination of this function θ involves only ω and hence has a meaning for the A-surfaces. If such a family of curves is taken upon any A-surface S and the corresponding envelope of the plane, drawn as in the transformation of pseudospherical surfaces, is determined, it is found that it is an A-surface whose lines of curvature are in correspondence with the lines of curvature upon S and have the spherical representation whose linear element is given by (a) when ω is replaced by θ. Since θ involves an arbitrary constant, there is an infinity of transforms of each A-surface. In turn, each of these surfaces can be transformed in an infinity of ways, and we shall see later that the functions giving the transformation are obtained by algebraic operations and quadratures.

The development of the preceding theory is contained in §4, and in §5 we consider some of its results. It is found that only in the case of pseudospherical surfaces does the point of tangency of the envelope fall on the intersection of the two planes, so that the curves determined by the angle θ are geodesics only in this case. We then investigate the problem of finding when the lines joining corresponding points on a surface and its transform constitute a normal congruence. We have seen that through each point M of an A-surface S an infinity of planes can be drawn which envelop the transforms of S. We find that all the points of contact are on a circle whose axis is the normal to S at M, which gives an interesting interpretation of the cyclic systems of circles associated with every A-surface.

Bianchi has considered the group of surfaces which have the following property: *The sphere described on every segment of the normal comprised between the two centers of curvature as diameters, cut a fixed sphere in great circles, or orthogonally, or pass through the center.** To these three types he has given the respective names, *elliptic*, *hyperbolic* and *parabolic*. We shall refer to them as the *surfaces of Bianchi*. After having derived in §6 the expressions for the coordinates of these surfaces in a manner similar to that followed by Bianchi and noted that they are A-surfaces, we determine their transforms. These are shown to be surfaces of Bianchi of the parabolic type.

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* Nuove ricerche sulle superficie pseudosferiche, Annali di Matematica, Ser. 2, Vol. 24, p. 347.

In §7, we discover generalized transformations of A-surfaces suggested by the generalization by Bäcklund of the transformations of Bianchi for pseudospherical surfaces. The transformations of Bäcklund may be defined as follows: Upon a pseudospherical surface S there is an infinity of families of curves which are such that, if planes be drawn making a certain constant angle σ with the tangent planes to S, passing through the points of contact and intersecting the latter planes in the tangent lines to the curves of one of these families, these planes envelop a surface of the same total curvature as S. These curves may be determined by the angle θ which their tangents make with the tangents to the lines of curvature $v =$ const., and the complete determination of θ requires the solution of a Riccati equation whose coefficients involve σ and ω, where the latter is the function entering in the spherical representation (a) of S. From this it follows that θ can be used in connection with all A-surfaces having the same representation. If, for all of these surfaces, we construct planes as in the case of the pseudospherical surface of the same group, their envelopes are A-surfaces and all of them form a group. In this manner, after the integration of a Riccati equation, we can get, from one surface S, a doubly infinite system of A-surfaces, for σ and the constant of integration are arbitrary. Each of these surfaces belongs to a different group of surfaces, and the complete integration of a partial differential equation of the second order puts us in a position for finding all the surfaces of all these groups by direct operations.

Bianchi has shown,* that if a pseudospherical surface S be transformed into S_1 and S_2 so that the angles between the tangent planes to S and to the latter shall be respectively σ_1 and σ_2, there is known a function ϕ by means of which S_1 and S_2 can be transformed into the same surface S_3 so that the angles between the tangent planes to S_3 and to S_1 and S_2 respectively are σ_2 and σ_1. We show that this theorem is true for all A-surfaces, in consequence of which we have that when one knows how to transform an A-surface in a general manner, the transformations of the transforms can be effected by algebraic processes and differentiation.

General transformations of surfaces of Bianchi are considered in §9, and it is found that surfaces of the parabolic type and these alone are transformed into surfaces of this kind whatever be σ.

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* Lezioni, p. 435; Ger. trans., p. 461.

§10 is devoted to the study of surfaces corresponding to the solution $\omega = 0$ of equation (b) and their transforms. It closes with an investigation of the transforms of the moulure surfaces.

I.

§1.—*A-Surfaces. Definition and Determination.*

Consider a surface S upon which the lines of curvature are parametric and refer it to the trihedron whose x-, y-, z-axes are respectively the tangents to the curves v=const., u=const. at the point and the normal to the surface; the positive directions along these lines are so chosen that as one looks upon the tangent plane from a point on the positive part of the z-axis, the positive y-axis is to the left of the positive x-axis. If we write the linear elements of the surface and its spherical representation in the forms

$$ds^2 = A^2du^2 + C^2dv^2 \tag{1}$$

and

$$d\sigma^2 = q^2du^2 + p_1^2dv^2, \tag{2}$$

and write further

$$\rho_{gu} = A/r, \quad \rho_{gv} = C/r_1, \tag{3}$$

where ρ_{gu} and ρ_{gv} denote the geodesic curvature of the curves $v =$ const. and $u =$ const. respectively, one can find the following necessary and sufficient relations between these several functions:*

$$\frac{\partial p_1}{\partial u} = -qr_1, \quad \frac{\partial q}{\partial v} = rp_1, \quad \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -qp_1,$$
$$r = -\frac{1}{C}\frac{\partial A}{\partial v}, \quad r_1 = \frac{1}{A}\frac{\partial C}{\partial u}.$$

We consider the surfaces for which

$$d\sigma^2 = \sin^2\omega\, du^2 + \cos^2\omega\, dv^2. \tag{4}$$

* Darboux, Leçons, Vol. 2, pp. 386, 392.

If we substitute these values for p_1 and q in the above formulæ, we have the following fundamental relations:

$$p_1 = \cos\omega, \quad q = \sin\omega, \quad r = \frac{\partial\omega}{\partial v}, \quad r_1 = \frac{\partial\omega}{\partial u}, \tag{5}$$

$$-\frac{1}{C}\frac{\partial A}{\partial v} = \frac{\partial\omega}{\partial v}, \quad \frac{1}{A}\frac{\partial C}{\partial v} = \frac{\partial\omega}{\partial u} \tag{6}$$

and

$$\frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2} = \sin\omega\cos\omega. \tag{7}$$

From the preceding, it follows that every solution of this equation leads to as many surfaces of this kind as there are solutions A and C of the corresponding equations (6). From (4) it follows that all the surfaces arising from the same function ω have the same spherical representation of their lines of curvature; hence, the tangent planes at corresponding points of all these surfaces are parallel and the tangents to homologous lines of curvature at these points are parallel. Moreover, from the manner in which we defined these surfaces, it follows that every surface which has its lines of curvature homologous to the similar lines on an *A-surface* and at the same time corresponds with parallelism of tangent planes, is an A-surface.

Suppose that we have given two A-surfaces S_1 and S_2, referred to their lines of curvature. Denoting by W_1 and W_2 the distances from the origin upon the tangent planes to S_1 and S_2 at corresponding points, we have that W_1 and W_2 are particular solutions of the equation*

$$\frac{\partial^2\psi}{\partial u\,\partial v} - \frac{\partial \log\sin\omega}{\partial v}\frac{\partial\psi}{\partial u} - \frac{\partial\log\cos\omega}{\partial u}\frac{\partial\psi}{\partial v} = 0, \tag{8}$$

and the point-coordinates of these surfaces are given by replacing ψ by W_1 and W_2 respectively, in the formulæ*

$$x = \psi X + \Delta(\psi, X), \quad y = \psi Y + \Delta(\psi, Y), \quad z = \psi Z + \Delta(\psi, Z), \tag{9}$$

* Bianchi, Lezioni, p. 137 ; Ger. trans., p. 140.

where X, Y, Z denote the direction-cosines of the normal to the surface with respect to fixed axes, and $\Delta(\psi, \theta)$ is the mixed differential parameter formed with respect to the quadratic form (4). Since the tangent planes to these surfaces are parallel, it follows that the difference between W_2 and W_1, denoted by λ, is a solution of equation (8). If, then, we have an A-surface with rectangular coordinates, x, y, z, referred to fixed axes and we can find a solution of (8), we have directly a second A-surface defined by

$$\xi = x + X\lambda + \Delta(\lambda, X), \quad \eta = y + Y\lambda + \Delta(\lambda, Y), \quad \zeta = z + Z\lambda + \Delta(\lambda, Z), \tag{10}$$

for X, Y, Z are known in this case.

In particular, equation (8) is satisfied by a constant. Then (10) defines a parallel surface, so that we have the evident theorem: *All the parallel surfaces of an A-surface are A-surfaces.*

Since the tangents to homologous lines of curvature at corresponding points on two A-surfaces are parallel, we have

$$\frac{\partial x_2}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \quad \frac{\partial x_2}{\partial v} = \mu \frac{\partial x_1}{\partial v}, \tag{11}$$

and similar relations in y and z. From these we get

$$A_2 = \lambda A_1, \quad C_2 = \mu C_1.$$

From (6) it follows that

$$\frac{1}{C_1}\frac{\partial A_1}{\partial v} = \frac{1}{C_2}\frac{\partial A_2}{\partial v}, \quad \frac{1}{A_1}\frac{\partial C_1}{\partial u} = \frac{1}{A_2}\frac{\partial C_2}{\partial u},$$

which, by means of the above, can be replaced by

$$\left.\begin{aligned} \frac{\partial \lambda}{\partial v} + (\lambda - \mu)\frac{\partial \log A_1}{\partial v} &= 0, \\ \frac{\partial \mu}{\partial u} + (\mu - \lambda)\frac{\partial \log C_1}{\partial u} &= 0. \end{aligned}\right\} \tag{12}$$

Hence, when an A-surface, S_1, is given, the complete determination of all surfaces with the same spherical representation of their lines of curvature requires the solution of the system (12) and quadratures (11).

The system (12) can be replaced by a slightly different one in which the coefficients involve ω directly. Thus, in consequence of the Rodrigues formulæ,*

$$\frac{\partial x_1}{\partial u} = \rho_1 \frac{\partial X}{\partial u}, \quad \frac{\partial x_1}{\partial v} = \rho_2 \frac{\partial X}{\partial v},$$

where ρ_1 and ρ_2 denote the principal radii of curvature of S_1, we have

$$A_2 = \frac{\rho_1'}{\rho_1} A_1, \quad C_2 = \frac{\rho_2'}{\rho_2} C_1,$$

where ρ_1', ρ_2' are the radii of curvature for S_2. But †

$$\rho_1 = -\frac{A}{q}, \quad \rho_2 = \frac{C}{p_1}, \tag{13}$$

so that the above reduce to

$$A_2 = -l \sin \omega, \quad C_2 = m \cos \omega,$$

where, for the moment, we have put l and m for ρ_1' and ρ_2'. If these values be substituted in the equations for S_2 of the form (6), one has for the determination of l and m,

$$\left.\begin{aligned} &\frac{\partial l}{\partial v} + (l - m)\frac{\partial \log \sin \omega}{\partial v} = 0, \\ &\frac{\partial m}{\partial u} + (m - l)\frac{\partial \log \cos \omega}{\partial u} = 0. \end{aligned}\right\} \tag{14}$$

When l and m are equal, they are constants, in which case S_2 is a sphere. If we make an exception of this evident solution and eliminate m from these equations and again l, we get

$$\left.\begin{aligned} &\frac{\partial^2 l}{\partial u \partial v} - \frac{\partial \log \sin \omega}{\partial v}\frac{\partial l}{\partial u} - \left[\frac{\partial}{\partial u}\log \cos \omega + \frac{\partial}{\partial u}\log\left(\frac{\partial \log \sin \omega}{\partial v}\right)\right]\frac{\partial l}{\partial v} = 0, \\ &\frac{\partial^2 m}{\partial u \partial v} - \left[\frac{\partial}{\partial v}\log \sin \omega + \frac{\partial}{\partial v}\log\left(\frac{\partial \log \cos \omega}{\partial u}\right)\right]\frac{\partial m}{\partial u} \\ &\qquad\qquad\qquad - \frac{\partial}{\partial u}\log \cos \omega \frac{\partial m}{\partial v} = 0. \end{aligned}\right\} \tag{15}$$

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* Bianchi, Lezioni, p. 101; Ger. trans., p. 102.

† Darboux, Leçons, Vol. 2, p. 392.

When a solution of either of these equations is obtained, the other function corresponding is found from (14) by two quadratures.

All of these processes for the determination of all surfaces with the same representation of their lines of curvature as a given A-surface come back to the solution of a partial differential equation of the second order. But the subsequent steps in the first introduce the least difficulty, and, consequently, we are inclined to give preference to it.

§2.—*Particular A-surfaces.*

The necessary and sufficient condition that the two equations (15) be the same, in which case they have the form (8), is that the two equations

$$\frac{\partial^2}{\partial u\, \partial v}\log \sin \omega = 0, \quad \frac{\partial^2}{\partial u\, \partial v}\log \cos \omega = 0,$$

be satisfied. From these it follows that

$$\sin^2 \omega = U_1 V_1, \quad \cos^2 \omega = U_2 V_2,$$

where U_1, U_2 are functions of u alone and V_1, V_2 functions of v alone. These functions must satisfy the condition

$$U_1 V_1 + U_2 V_2 = 1, \tag{16}$$

which, by differentiation with respect to u, becomes

$$U_1' V_1 + U_2' V_2 = 0,$$

from which it follows that either U_1 and U_2 are constants, or

$$U_1 = c\,U_2 + d, \quad V_2 = -c\,V_1,$$

where c and d are constants. When these values are substituted in equation (16), one finds that V_1 and V_2 are constants. Hence, the above conditions are satisfied only in case ω is a function of a single variable, say u. Then, from (14), one has that l is a function of u alone and m is given by two quadratures,

so that all these surfaces are given by quadratures. Let us consider them more minutely.

Since ω is a function of u alone, we have from (6) that A is also a function of u alone, and, consequently, the lines of curvature $v = \text{const.}$ are geodesics.* They, therefore, belong to the class of surfaces, considered by Monge,† whose lines of curvature in one system are plane and the surface is generated by this curve when its plane envelops a developable surface. Again, the coefficients in (4) are functions of u alone, so that the lines of curvature are represented on the sphere by a system of great circles with a common diameter. Hence, the generating developable is cylindrical, so that the surfaces are so-called *moulure* surfaces.

Now, equation (7) reduces to

$$\frac{d^2\omega}{du^2} = \sin\omega\cos\omega,$$

from which, by integration, we have

$$\left(\frac{d\omega}{du}\right)^2 = \sin^2\omega + a, \tag{17}$$

where a is an arbitrary constant. In general, the integral of (17) is an elliptic function of u, but when a is zero, the integral is

$$\tan\frac{\omega}{2} = ce^u. \tag{18}$$

Now equation (8) can be integrated; the complete integral is

$$\psi = \cos\omega\, U + V,$$

where U and V are arbitrary functions of u and v respectively. For this case, X, Y, Z are given by quadratures, and after these have been determined, all the corresponding moulure surfaces are given directly by formulæ (9). The pseudo-spherical surfaces with this representation of their lines of curvature are surfaces

* Bianchi, Lezioni, p. 144; Ger. trans., p. 148.

† Application de l'Analyse à la Geométrie, 5th ed., p. 322.

of revolution, for their linear element is of the form (21). We shall return to a consideration of these surfaces.

Let us inquire now in what cases A and C, as given by (6), are functions of ω alone. Put

$$A = \phi(\omega), \quad C = \psi(\omega),$$

then equations (6) become

$$\phi' + \psi = 0, \quad \psi' - \phi = 0.$$

Eliminating ψ, we get

$$\phi'' + \phi = 0,$$

so that

$$\left.\begin{aligned} A &= c_1 \cos\omega + c_2 \sin\omega, \\ C &= c_1 \sin\omega - c_2 \cos\omega, \end{aligned}\right\} \tag{19}$$

where c_1 and c_2 are arbitrary constants. If, in particular, we establish between these constants a relation such that we can put

$$c_1 = \cos\alpha, \quad c_2 = \sin\alpha,$$

where α is a constant (and thus consider a sub-class of the surfaces with the values (19)), we have

$$A = \cos(\omega - \alpha), \quad C = \sin(\omega - \alpha).$$

Hence, among all the surfaces of Bonnet arising from a solution ω of equation (7), there is a family whose linear element has the form

$$ds^2 = \cos^2(\omega - \alpha)\, du^2 + \sin^2(\omega - \alpha)\, dv^2, \tag{20}$$

where α is the parameter of the family. It is important to note that all the surfaces with this linear element belong uniquely to the class with the function ω, for, when ω is a solution of equation (7), $\omega - \alpha$ cannot be an integral for any value of α except zero. When α is zero, the above linear element becomes

$$ds^2 = \cos^2\omega\, du^2 + \sin^2\omega\, dv^2. \tag{21}$$

But this is the well-known characteristic form of the linear element of a pseudospherical surface of curvature minus one referred to its lines of curvature.

It is readily seen that all the parallels of pseudospherical surfaces belong to the class for which A and C have the forms (19). For, if the constant distance be denoted by a, the following relations obtain among the fundamental coefficients E, F, G; L, M, N of a surface and the similar functions E_1, F_1, G_1; L_1, M_1, N_1 of its parallel:*

$$\left.\begin{aligned} E_1 &= E\left(1-\frac{a^2}{\rho_1\rho_2}\right) + aL\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right], \\ F_1 &= F\left(1-\frac{a^2}{\rho_1\rho_2}\right) + aM\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right], \\ G_1 &= G\left(1-\frac{a^2}{\rho_1\rho_2}\right) + aN\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right]. \end{aligned}\right\} \tag{22}$$

For the pseudospherical surface referred to its lines of curvature, these quantities have the forms

$$\begin{aligned} &E = A^2 = \cos^2\omega, \qquad F = 0, \quad G = C^2 = \sin^2\omega, \\ &L = \frac{A^2}{\rho_1} = \sin\omega\cos\omega, \quad M = 0, \quad N = \frac{C^2}{\rho_2} = -\sin\omega\cos\omega. \end{aligned}$$

From these we get by a ready calculation

$$E_1 = A_1^2 = (\cos\omega - a\sin\omega)^2, \qquad G_1 = C_1^2 = (\sin\omega + a\cos\omega)^2, \tag{23}$$

which are evidently of the form (19). These expressions for A_1 and C_1 differ from those of (19) by the same constant factor, so that we have the theorem:

The only A-surfaces for which A and C are functions of ω *alone are pseudospherical surfaces, their parallels and the homothetics of these.*

In order to find the A-surfaces whose principal radii are functions of one another, it is only necessary to recall the theorem of Weingarten* that this problem reduces to the determination of orthogonal systems on the sphere such that the corresponding linear element can be given the form

$$d\sigma^2 = \frac{du^2}{x^2} + \frac{dv^2}{\phi'^2(x)}, \tag{24}$$

* Knoblauch, Einleitung in die Allgemeine Theorie der Krummen Flächen, p. 285.

where x is a function of u and v, and that when such a system is found, the principal radii of the surface with this representation of its lines of curvature are given by

$$\rho_1 = \phi(x), \quad \rho_2 = \phi(x) - x\phi'(x), \tag{25}$$

where the prime denotes differentiation. Comparing (24) with (4), one finds

$$\rho_1 = -\tan\omega + c, \quad \rho_2 = \cot\alpha + c, \tag{26}$$

where c is a constant of integration. When c is zero, the total curvature of the surface is minus one, and for other values of c the surfaces are the parallels of the former. Hence the theorem:

The pseudospherical surfaces and their parallels are the only Weingarten surfaces which are A-surfaces.

§3.—*Lie Transformations of A-Surfaces.*

Thus far we have considered only the lines of curvature as parametric. If, now, we effect the change of parameters given by

$$\alpha = \frac{u+v}{2}, \quad \beta = \frac{u-v}{2}, \tag{27}$$

the equation (7) takes the form

$$\frac{\partial^2\omega}{\partial\alpha\,\partial\beta} = \sin\omega\cos\omega. \tag{28}$$

Lie† has remarked that if $\omega(\alpha, \beta)$ is a solution of this equation, so also is $\omega\left(m\alpha, \frac{\beta}{m}\right)$ for any value, except zero, of the constant m. If, then, $\omega(u, v)$

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* Darboux, Leçons, Vol. 3, p. 319.

† Ueber Flächen deren Krümmungsradien durch eine Relation verknüpft sind, Archiv for Mathematik og. Naturvidenskab, Vol. 4, p. 510; also Darboux, l. c., p. 381.

is a solution of equation (7) and, consequently, $\omega(\alpha+\beta,\, \alpha-\beta)$ of equation (28), it follows that $\omega\left(\alpha m+\frac{\beta}{m},\, \alpha m-\frac{\beta}{m}\right)$ is a solution of the latter equation and, consequently,

$$\omega\left[\frac{(m^2+1)u+(m^2-1)v}{2m},\ \frac{(m^2-1)u+(m^2+1)v}{2m}\right]$$

is a solution of (7). If we put $\frac{1-\cos\sigma}{\sin\sigma}$ in place of m, this takes the much simpler form

$$\omega\left(\frac{u-v\cos\sigma}{\sin\sigma},\ \frac{v-u\cos\sigma}{\sin\sigma}\right), \tag{29}$$

where evidently σ is a constant taking any values but 0 and π.

When this observation of Lie is considered in connection with the pseudospherical surfaces arising from the integration of equation (7), it is seen that the knowledge of one surface of this kind leads to an infinity of them. But this so-called *transformation of Lie* is equally applicable to any surface of Bonnet and leads to a more general result. Thus, if we have given any surface of Bonnet referred to its lines of curvature, and the linear element of the sphere in the form (4), we have the spherical representation of an infinity of groups of these surfaces, and the determination of them all requires the integration of equation (8), in which ω has been given the general form (29).

It is evident that when the given surface is a parallel of a pseudospherical surface, that is $A=\phi(\omega)$ and $C=\psi(\omega)$, the linear element of a Lie transform is obtained when u and v in A and C are replaced by $\frac{u-v\cos\sigma}{\sin\sigma}$ and $\frac{v-u\cos\sigma}{\sin\sigma}$ respectively. We inquire for all cases of this kind. If the above expressions be denoted by u_1 and v_1, we must have from (6)

$$-\frac{1}{C(u_1,v_1)}\frac{\partial A(u_1,v_1)}{\partial v}=\frac{\partial\omega(u_1,v_1)}{\partial v},\quad \frac{1}{A(u_1,v_1)}\frac{\partial C(u_1,v_1)}{\partial u}=\frac{\partial\omega(u_1,v_1)}{\partial u}, \tag{30}$$

of which the first reduces to

$$\frac{1}{C(u_1,v_1)}\left[\frac{\partial A(u_1,v_1)}{\partial u}\cot\sigma-\frac{1}{\sin\sigma}\frac{\partial A(u_1,v_1)}{\partial v_1}\right]$$
$$=-\frac{\partial\omega(u_1,v_1)}{\partial u_1}\cot\sigma+\frac{\partial\omega(u_1,v_1)}{\partial v_1}\frac{1}{\sin\sigma}. \tag{31}$$

But
$$-\frac{1}{C(u_1, v_1)}\frac{\partial A(u_1, v_1)}{\partial v_1} = \frac{\partial \omega(u_1, v_1)}{\partial v_1}, \tag{32}$$

so that the above reduces to

$$-\frac{1}{C(u_1, v_1)}\frac{\partial A(u_1, v_1)}{\partial u_1} = \frac{\partial \omega(u_1, v_1)}{\partial u_1}. \tag{33}$$

These two relations show that A is a function of ω; similarly, C is a function of ω. Hence, only for pseudospherical surfaces and their parallels does the above change of parameters give the linear element of the new surface.

If in the transformation (29) we replace σ by $\pi - \sigma$, it becomes

$$\omega\left(\frac{u + v\cos\sigma}{\sin\sigma}, \quad \frac{v + u\cos\sigma}{\sin\sigma}\right). \tag{34}$$

When the transformations indicated by (29) and (34) are applied consecutively to a function $\omega(u, v)$, one gets the latter again. Hence, if the Lie transformation of angle σ be denoted by L_σ, this result may be indicated by

$$L_\sigma^{-1} = L_{\pi - \sigma}. \tag{35}$$

§4.— *Complementary Transformations of A-Surfaces.*

We are now going to establish for A-surfaces transformations analogous to the *transformations of Bianchi* of pseudospherical surfaces. We begin with a short development of the latter.*

Consider a pseudospherical surface S referred to the moving trihedron previously defined and whose fundamental functions and their relations are given by (21), (4), (5), (6) and (7). Through the point M and in the tangent plane we draw a line making an angle θ with the x-axis. Denote by M_1 the point on this line at unit distance from M. Its rectangular coordinates with respect to the trihedron are evidently

$$\cos\theta, \quad \sin\theta, \quad 0.$$

* Darboux, Leçons, Vol. 3, p. 426.

When M moves over S, the projections of the displacement of M_1 on these axes are*

$$\left.\begin{aligned}
&-\sin\theta d\theta + \cos\omega du - \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\sin\theta,\\
&\cos\theta d\theta + \sin\omega dv + \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\cos\theta,\\
&\cos\omega\sin\theta dv - \sin\omega\cos\theta du.
\end{aligned}\right\} \tag{36}$$

We consider now the locus of the point M_1 which we denote by S_1, and we demand the conditions which θ must satisfy in order that the line MM_1 be tangent to S_1 at M_1 and the tangent plane to S_1 be perpendicular to the tangent plane to S. Then the direction-cosines of the tangent plane to S_1 at M_1 are

$$+\sin\theta, \quad -\cos\theta, \quad 0, \tag{37}$$

and since the tangent to the above displacement must be in this plane, we must have

$$d\theta + \left(\frac{\partial\omega}{\partial v} - \sin\theta\cos\omega\right)du + \left(\frac{\partial\omega}{\partial u} + \sin\omega\cos\theta\right)dv = 0.$$

Equating to zero the coefficients of du and dv, we have, for the determination of θ, the two equations

$$\left.\begin{aligned}
\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v} &= \cos\omega\sin\theta,\\
\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u} &= -\sin\omega\cos\theta.
\end{aligned}\right\} \tag{38}$$

If θ be eliminated from these equations, we are brought to equation (7), so that these equations are compatible. Moreover, if ω be eliminated, it is found that θ satisfies equation (7).

By means of (38) the above expressions for the projections of the displacement of M_1 can be put in the form

$$\left\{\begin{aligned}
&\cos\theta(\cos\omega\cos\theta du + \sin\omega\sin\theta dv),\\
&\sin\theta(\cos\omega\cos\theta du + \sin\omega\sin\theta dv),\\
&\qquad\cos\omega\sin\theta dv - \sin\omega\cos\theta du,
\end{aligned}\right.$$

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* Ib., Vol. 2, p. 385.

from which it follows that the linear element of S_1 is

$$ds_1^2 = \cos^2\theta du^2 + \sin^2\theta dv^2.$$

Since θ is a solution of equation (7), it follows that S_1 is a pseudospherical surface. Hence, if we draw through a point of a pseudospherical surface of unit curvature a line in the tangent plane and making an angle θ, given by (38), with the tangent to the line of curvature $v = \text{const.}$, the locus of the point at unit distance on this line is a pseudospherical surface whose tangent plane is perpendicular to the corresponding tangent plane to S. Darboux calls this the *transformation of Bianchi*. Since the system (38) can be replaced by a Riccati equation,* it follows that θ involves an arbitrary constant, and, consequently, there is an infinity of such transforms.

The surface S_1, which is the transform of S corresponding to a given angle θ, may be defined, in another manner, as the envelope of the plane through M perpendicular to the tangent plane to M and meeting the latter in a line which makes an angle θ with the tangent to the line of curvature, $v = \text{const.}$ We shall consider the character of this envelope when S is any A-surface. The coordinates of the point M_1, where this plane touches the envelope, are evidently of the form

$$\lambda\cos\theta, \quad \lambda\sin\theta, \quad \mu,$$

where λ and μ are functions of u and v, which have to be determined.

The projections on the moving axes of a displacement of M_1 have the following expressions: †

$$\left.\begin{aligned}
d\xi &= \cos\theta d\lambda - \lambda\sin\theta d\theta + Adu + \mu\sin\omega du - \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\lambda\sin\theta, \\
d\eta &= \sin\theta d\lambda + \lambda\cos\theta d\theta + Cdv + \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\lambda\cos\theta - \mu\cos\omega dv, \\
d\zeta &= d\mu + \lambda\cos\omega dv - \lambda\sin\omega\cos\theta du.
\end{aligned}\right\} \quad (40')$$

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* Bianchi, Lezioni, p. 429; Germ. trans., p. 454.

† Darboux, Leçons, Vol. 2, p. 385.

Since θ is to be a solution of the system (38), these expressions can be put in the form

$$\left.\begin{aligned} d\xi &= \left(\cos\theta\frac{\partial\lambda}{\partial u}+A+\mu\sin\omega-\lambda\sin^2\theta\cos\omega\right)du \\ &\qquad\qquad+\left(\cos\theta\frac{\partial\lambda}{\partial v}+\lambda\sin\omega\sin\theta\cos\theta\right)dv, \\ d\eta &= \left(\sin\theta\frac{\partial\lambda}{\partial u}+\lambda\cos\theta\sin\theta\cos\omega\right)du \\ &\qquad\qquad+\left(\sin\theta\frac{\partial\lambda}{\partial v}+C-\mu\cos\omega-\lambda\cos^2\theta\sin\omega\right)dv, \\ d\zeta &= \left(\frac{\partial\mu}{\partial u}-\lambda\sin\omega\cos\theta\right)du+\left(\frac{\partial\mu}{\partial v}+\lambda\cos\omega\sin\theta\right)dv. \end{aligned}\right\} \quad (40)$$

From the expressions (37) for the direction-cosines of the tangent plane to S_1, it follows that the above values must satisfy the equation

$$\sin\theta d\xi-\cos\theta d\eta=0,$$

which by (40) reduces to

$$\sin\theta\,(\lambda\cos\omega-\mu\sin\omega-A)\,du+\cos\theta\,(C-\lambda\sin\omega-\mu\cos\omega)\,dv=0.$$

Equating to zero the coefficients of du and dv, we have

$$\left.\begin{aligned} A&=\lambda\cos\omega-\mu\sin\omega, \\ C&=\lambda\sin\omega+\mu\cos\omega, \end{aligned}\right\} \quad (41)$$

from which it follows that

$$\left.\begin{aligned} \lambda&=\quad A\cos\omega+C\sin\omega, \\ \mu&=-A\sin\omega+C\cos\omega. \end{aligned}\right\} \quad (42)$$

Substituting these values for λ and μ in (40), the latter become

$$\left.\begin{aligned} d\xi &= \cos\theta\left[\cos\omega\frac{\partial A}{\partial u}+\sin\omega\frac{\partial C}{\partial u}+\mu\frac{\partial\omega}{\partial u}+\lambda\cos\theta\cos\omega\right]du \\ &\qquad+\cos\theta\left[\cos\omega\frac{\partial A}{\partial v}+\sin\omega\frac{\partial C}{\partial v}+\mu\frac{\partial\omega}{\partial v}+\lambda\sin\theta\sin\omega\right]dv, \\ d\eta &= \sin\theta\left[\cos\omega\frac{\partial A}{\partial u}+\sin\omega\frac{\partial C}{\partial u}+\mu\frac{\partial\omega}{\partial u}+\lambda\cos\theta\cos\omega\right]du \\ &\qquad+\sin\theta\left[\cos\omega\frac{\partial A}{\partial v}+\sin\omega\frac{\partial C}{\partial v}+\mu\frac{\partial\omega}{\partial v}+\lambda\sin\theta\sin\omega\right]dv, \\ d\zeta &= \left[-\sin\omega\frac{\partial A}{\partial u}+\cos\omega\frac{\partial C}{\partial u}-\lambda\frac{\partial\omega}{\partial u}-\lambda\cos\theta\sin\omega\right]du \\ &\qquad+\left[-\sin\omega\frac{\partial A}{\partial v}+\cos\omega\frac{\partial C}{\partial v}-\lambda\frac{\partial\omega}{\partial v}+\lambda\cos\omega\sin\theta\right]dv. \end{aligned}\right\} \quad (43)$$

From this we have for the linear element of S_1,

$$ds_1^2 = \left(\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} + \lambda \cos\theta\right)^2 du^2 + \left(\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} + \lambda \sin\theta\right)^2 dv^2. \quad (44)$$

We are now in a position to show that the surface S_1 is an A-surface. It has been seen that all the surfaces of Bonnet corresponding to the same function ω have parallel tangent planes and their lines of curvature in correspondence. Let Σ denote the pseudospherical surface related in this manner to our original surface S, and m denote the point on the former corresponding to M on the latter. Since the trihedrons at M and m are parallel, the plane through m perpendicular to the tangent plane at m and forming an angle θ with the x-axis, envelopes the pseudospherical surface Σ_1, which is the Bianchi transform of Σ for the angle θ. Hence S_1 and Σ_1 correspond with parallelism of tangent planes. Again, the parametric curves on S_1 correspond to the lines of curvature on S, consequently to the similar lines on Σ and, therefore, to the lines of curvature on Σ_1. Hence, the parametric lines on S_1, which form an orthogonal system (44), are represented on the sphere by such a system, so that these lines are lines of curvature for S_1. Thus S_1 corresponds with Σ_1 with parallelism of tangent planes and lines of curvature, and, therefore, is an A-surface.

If we write (44) in the form

$$ds_1^2 = A_1^2\, du^2 + C_1^2\, dv^2,$$

it is readily shown by (6) and (38) that

$$-\frac{1}{C_1}\frac{\partial A_1}{\partial v} = \frac{\partial \theta}{\partial v} \qquad \frac{1}{A_1}\frac{\partial C_1}{\partial u} = \frac{\partial \theta}{\partial u},$$

which verifies the above geometrical reasoning. Hence:

Given an A-surface corresponding to a solution ω of equation (7). *If planes be drawn perpendicular to the tangent planes through the point of contact and making an angle θ, given by* (38), *with the tangents to the curves $v = const.$, they envelop a new A-surface corresponding to the solution θ of equation* (7).

In order to find all the transform of S we must have the general integral θ of the system (38). If θ_1 denote any particular integral, the general integral

θ is given by *

$$\cot\left(\frac{\theta-\theta_1}{2}\right)=\beta_1 e^{-\alpha_1}, \tag{45}$$

where α_1 and β_1 are given by the quadratures

$$\left.\begin{aligned} d\alpha_1 &= \cos\theta_1 \cos\omega\, du + \sin\theta_1 \sin\omega\, dv, \\ d\beta_1 &= e^{\alpha_1}(\sin\theta_1 \cos\omega\, du - \cos\theta_1 \sin\omega\, dv)\,; \end{aligned}\right\} \tag{46}$$

the right-hand members are readily found to be exact differentials when it is noted that θ_1 satisfies (38). Since these quadratures are general, it follows that θ involves an arbitrary constant so that there is an infinity of transformations of S. But each of these transforms is only one of a group of A-surfaces with the same spherical representation of the lines of curvature. From (8) it follows that the complete determination of all these surfaces arising from a single A-surface requires the integration of the equation

$$\frac{\partial^2\psi}{\partial u\,\partial v}-\frac{\partial \log\sin\theta}{\partial v}\frac{\partial\psi}{\partial u}-\frac{\partial\log\cos\theta}{\partial u}\frac{\partial\psi}{\partial v}=0. \tag{47}$$

It is evident that θ, as given by (45), is the same for all A-surfaces belonging to the same function ω. Hence, when all the surfaces arising from the general integral of (47) are determined, we have among them the transforms of all the surfaces with the same spherical representation as S.

Proceeding as in the case of S, we have that the transformation of the transforms of S and of all A-surfaces belonging to the same ω requires the solution of the system

$$\left.\begin{aligned} \frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial v} &= \cos\theta\sin\phi, \\ \frac{\partial\phi}{\partial v}+\frac{\partial\theta}{\partial u} &= -\sin\theta\cos\phi. \end{aligned}\right\} \tag{48}$$

Now $\phi_1=\omega+\pi$ is a solution of these equations, hence the general integral for surfaces corresponding to the angle θ_1 is given by †

$$\cot\left(\frac{\phi-\phi_1}{2}\right)=-\gamma_1 e^{\alpha_1}, \tag{49}$$

* Darboux, Leçons, Vol. 3, p. 458.

† Darboux, l. c.

where α_1 is given by (46) and γ_1 by the quadrature

$$d\gamma_1 = e^{-\alpha_1}(\cos\theta_1 \sin\omega\, du - \sin\theta_1 \cos\omega\, dv). \tag{50}$$

Thus, by a quadrature, we have the general function ϕ for the transformation of surfaces which are transforms by θ_1 of S and the surfaces of its class.

The formula (45) can also be written

$$\cot\left(\frac{\theta - \theta_1}{2}\right) = -\beta e^{-\alpha},$$

where α and β are given by (46) when θ_1 is replaced by θ. Between these functions exist the relations

$$e^{\alpha} = \frac{e^{\alpha_1}}{e^{2\alpha_1} + \beta_1^2}, \quad \beta = \frac{-\beta_1}{e^{2\alpha_1} + \beta_1^2},$$

so that α and β are found by algebraic processes. Again, formula (49) can be written

$$\cot\left(\frac{\phi - \phi_1}{2}\right) = \gamma e^{\alpha}, \tag{49'}$$

where γ is given by the quadrature

$$d\gamma = e^{-\alpha}(\cot\theta \sin\omega\, du - \sin\theta \cos\omega\, dv).$$

Hence, it requires this further quadrature to obtain the function ϕ giving the transformations of all the transforms of S and the surfaces with the same representation of the lines of curvature. And all the other surfaces of this third general group of surfaces follow from the integration of an equation of the form (8) in which ω has been replaced by the general solution of (49′), which involves at least two arbitrary constants. We shall consider further transformations later.

§5.—*Particular Transformations. Associated Cyclic Systems.*

We shall consider now the expressions for λ and μ. For λ to be zero, we must have

$$A = l\sin\omega, \quad C = -l\cos\omega,$$

where l is an auxiliary function. From (6) we find that l is a constant, and, consequently, from (4) and (13), the corresponding surface is a sphere.

When S is pseudospherical, λ is unity and μ is zero. For the latter to be satisfied, we must have

$$A = l \cos \omega, \qquad C = l \sin \omega,$$

where l is an auxiliary function. From (6) we have that l is a constant; hence, only in the case of pseudospherical surfaces is μ zero. For this case, S and S_1 are the focal nappes of a rectilinear congruence, which is normal, since the tangent planes are perpendicular. For none of the other A-surfaces do the lines of intersection of the tangent planes to S and S_1 form a normal congruence. Hence:

Upon the pseudospherical surfaces the curves $\theta = const.$ are geodesics, and on no other A-surfaces.

We seek now all the cases for which λ is constant. If we substitute the values of A and C as given by (41) in (6) and consider λ constant, it is found that μ also is constant, unless ω is zero; similar results follow from the hypothesis that μ is constant. Now, the formulæ (41) take the form (19), so that *the only A-surfaces for which λ and μ can be constant are pseudospherical surfaces and their parallels, excepting in the case where ω is zero.**

This was an evident solution of the problem, for, from the manner in which parallel surfaces are associated with one another, it follows that a transform of angle θ for one is the same for all. Hence, if a denotes the distance between the tangent planes to a pseudospherical surface and a parallel, for the latter λ is the same as for the former and μ is plus or minus a according to the direction.

It has been seen that the transforms of the parallels of a pseudospherical surface are surfaces of this kind. We seek the general conditions which a surface S must satisfy in order the transforms are pseudospherical. From (39) and (44) it follows that we must have

$$\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} + (\lambda - \epsilon)\cos\theta = 0,$$
$$\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} + (\lambda - \epsilon)\sin\theta = 0,$$

* This will be discussed later.

where ε is plus or minus one. By means of (6) this can be written

$$\frac{\partial}{\partial u}(A^2 + C^2) = 2A(\varepsilon - \lambda)\cos\theta, \quad \frac{\partial}{\partial v}(A^2 + C^2) = 2C(\varepsilon - \lambda)\sin\theta. \quad (51)$$

It is readily found that these equations are compatible when θ is a solution of equation (38). Then, from these equations and (42) it is seen that a necessary condition is that given by

$$\frac{1}{A^2}\left[\frac{\partial}{\partial u}(A^2 + C^2)\right]^2 + \frac{1}{C^2}\left[\frac{\partial}{\partial v}(A^2 + C^2)\right]^2 = 4(A\cos\omega + C\sin\omega - \varepsilon)^2.$$

When this condition is satisfied and θ given by either of equations (51) satisfies equations (38), not only is the corresponding surface capable of such a transform but the transforming angle is given.

Equations (51) are satisfied identically when $A^2 + C^2$ is constant and λ is plus or minus one. For the former case we have

$$A = t\cos\sigma, \quad C = t\sin\sigma,$$

where t is constant and σ is an auxiliary angle. If these values be substituted in (6), it is found that

$$\sigma = \omega - \alpha,$$

where α is a constant. Hence for λ to be a constant and $A^2 + C^2$ to be constant, the surface S in both cases must be parallels of a pseudospherical surface. It is to be remarked that this is the only case where θ is not determined by equations (51). Hence, although there may be A-surfaces capable of a single pseudospherical transform, the pseudospherical surfaces and their parallels are the only surfaces which have an infinity of such transforms.

It has been noticed that the system (48), which gives the angle of transformation of the surfaces S_1, admits the solution $\phi_1 = \omega + \pi$. The corresponding transform of S_1 has for its spherical representation the linear element (4) and, therefore, belongs to the group of the original surface S. When the latter is pseudospherical, it coincides with this new surface S'. Moreover, this is true only for this case, for when μ is not zero, the tangent planes to S and S' are not coincident but parallel. Therefore, if S is an A-surface other than a pseudo-

spherical surface, we can get another of the same class as soon as we have a transform S_1 of S. With this same function θ we can continue to get A-surfaces belonging to the original group unless two of these surfaces S have the same transform, or we are brought to a pseudospherical surface or one of the surfaces reduces to a curve or point. Later, we shall have an example of this last exception.

As a verification of the fact the plane through M_1 and parallel to the tangent plane to S at M_1 envelops an A-surface of the same class as S, we have only to show that the value for μ given by (42) is a solution of equation (8). This is readily done in consequence of the relations (6).

We shall seek now for the coordinates, with respect to the trihedron for S, of the point of contact to the above envelope. These can be written

$$\lambda \cos\alpha, \quad \lambda \sin\alpha, \quad \mu,$$

where μ must have the value (42) and λ and α are to be determined. For this we make use of (40) and have only to express the condition that $d\zeta$ is zero. This gives the two equations

$$\left.\begin{aligned} \frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} + \lambda \cos\alpha = 0, \\ \frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} + \lambda \sin\alpha = 0, \end{aligned}\right\} \tag{51'}$$

which can be written

$$\frac{\partial}{\partial u}(A^2 + C^2) + 2\lambda A \cos\alpha = 0, \quad \frac{\partial}{\partial v}(A^2 + C^2) + 2\lambda C \sin\alpha = 0.$$

As in the case of equations (51), the above are compatible, so that we can determine λ and α at once, and then the surface completely.

The envelope of this plane is parallel to S when λ is zero, which is possible only in case $A^2 + C^2$ is constant, that is, only when S is pseudospherical or the parallel of such a surface. This could be seen also from the fact that μ must be zero, which leads to this particular case.

It has been seen that when S is a pseudospherical surface, the lines joining corresponding points on S and S_1 form a normal congruence. We inquire whether this is a general property for A-surfaces.

If we denote by ψ the angle which this line forms with its projection on the tangent plane to S, it follows from the expressions for λ and μ that

$$\sin\psi = \frac{-A\sin\omega + C\cos\omega}{\sqrt{A^2+C^2}}, \quad \cos\psi = \frac{A\cos\omega + C\sin\omega}{\sqrt{A^2+C^2}}. \tag{52}$$

Denote by ρ the distance, measured along this line, from M to the point m of intersection with a normal surface; then the coordinates of m with reference to the moving axes are

$$\rho\cos\psi\cos\theta, \quad \rho\cos\psi\sin\theta, \quad \rho\sin\psi.$$

The expressions for the displacements of m are readily found in a manner similar to (40). If we note that the direction-cosines of the line are

$$\cos\psi\cos\theta, \quad \cos\psi\sin\theta, \quad \sin\psi,$$

the condition that the displacement be normal to this line can be reduced to

$$d\rho + A\cos\theta\cos\psi du + C\sin\theta\cos\psi dv = 0,$$

which can be replaced by the two equations

$$\frac{\partial\rho}{\partial u} + A\cos\theta\cos\psi = 0, \quad \frac{\partial\rho}{\partial v} + C\sin\theta\cos\psi = 0.$$

Expressing the condition of integrability and in the reduction making use of (52) and (6) we get

$$(A\sin\omega - C\cos\omega)\Big[C^3\sin\theta\frac{\partial}{\partial u}\log\sqrt{A^2+C^2} \\ + A^3\cos\theta\frac{\partial}{\partial v}\log\sqrt{A^2+C^2} + AC\sin\theta\cos\theta\Big] = 0,$$

when the first factor is equated to zero, we get pseudospherical surfaces, as we have seen before. It is evident that the second factor doesn't vanish in general. But there are exceptional cases when it is satisfied. Thus, it will be found later that there are certain A-surfaces which admit a point for a transform, in which case all the above lines meet in this point and hence are cut at right angles by a family of concentric spheres.

It is important to notice that the expressions (42) for λ and μ do not involve θ. Hence, the points of the transforms of S corresponding to M lie on a circle of radius λ and whose plane is parallel to the tangent plane at M and at a distance μ from it. Moreover, this circle cuts the infinity of transforms of S orthogonally. Hence, these circles form a cyclic system and the axes of the circles are the normals to S. And it is well known that these normals form the only normal cyclic congruences.* Hence:

The circles of a cyclic system whose congruence of axes is normal to a family of parallel surfaces S are the loci of the points on the transforms of S corresponding to the points of intersections of the normals with S.

For the circles to be equal, λ must be constant. Preceding results enable us to state this theorem:†

When the circles of a cyclic system are equal and the congruence of axes is normal to a family of surfaces, the latter are the parallels of a pseudospherical surface. Moreover, the circles lie in the tangent planes to the latter, and have the points of contact for centers.

In consequence of a preceding result, we have the theorem:

The planes of the circles of a cyclic system, whose axes form a normal congruence, envelope an A-surface with the same representation of the lines of curvature as the surfaces orthogonal to the congruence. When these latter surfaces are the parallels of a pseudospherical surface, and only in this case, the above envelope belongs to this family of surfaces.

§6.—*Surfaces of Bianchi and their Transformations.*

In the introduction we have given Bianchi's definition of these surfaces and shall now determine their analytical expressions in the manner which he has suggested. With him we shall refer to them throughout the discussion as the surfaces Σ.

If we denote by p and q respectively, the distance from the origin upon the tangent plane and one-half the square of the distance to the point of contact,

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* Bianchi, Lezioni, p. 333; Ger. trans., p. 352.

† L. c., p. 332; Ger. trans., p. 351.

these surfaces are characterized by the equation*

$$2q + k - (\rho_1 + \rho_2)\, p + \rho_1\rho_2 = 0, \tag{53}$$

where ρ_1, ρ_2 are the principal radii of curvature and k is a constant, which is positive, negative or zero according as Σ is of the elliptic, hyperbolic or parabolic type. The corresponding fixed sphere has its center at the origin and its radius $\sqrt{\pm k}$. It is important to remark that all the parallels of a surface Σ are surfaces of Bianchi of the same type, since the centers of curvature are the same for all parallel surfaces.

Let S be the pseudospherical surface with the linear element (21). The direction-cosines of the tangents to the lines of curvature, denoted by X_1, Y_1, Z_1; X_2, Y_2, Z_2, have the following expressions:

$$X_1 = \frac{1}{\cos\omega}\,\frac{\partial x}{\partial u}, \quad X_2 = \frac{1}{\sin\omega}\,\frac{\partial x}{\partial v}, \tag{54}$$

and similarly for $Y_1, \ldots, Z_2$. Recalling our previous definition of the positive directions of these lines, we can deduce the following: †

$$\left.\begin{aligned}
&\frac{\partial X_1}{\partial u} = \frac{\partial\omega}{\partial v} X_2 - \sin\omega X, && \frac{\partial X_2}{\partial u} = -\frac{\partial\omega}{\partial v} X_1, && \frac{\partial X}{\partial u} = -\sin\omega X_1,\\
&\frac{\partial X_1}{\partial v} = \frac{\partial\omega}{\partial v} X_2, && \frac{\partial X_2}{\partial v} = -\frac{\partial\omega}{\partial u} X_1 + \cos\omega X, && \frac{\partial X}{\partial v} = -\cos\omega X_2,
\end{aligned}\right\} \tag{55}$$

and similarly for the Y's and Z's.

Denote by S_1' the transform by the particular integral θ_1 of the pseudospherical surface S with the linear element (21). Then the linear element of S_1' has the form

$$ds_1'^2 = \cos^2\theta_1\, du_2 + \sin^2\theta_1\, dv^2,$$

which, by means of (46) and (50), can be put in the form

$$ds_1'^2 = d\alpha_1^2 + e^{2\alpha_1} d\gamma_1^2.$$

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* L. c., p. 347.

† Bianchi, Lezioni, p. 94; Ger. trans., p. 94.

If, now, the method of Weingarten * for the determination of surfaces satisfying equations of the general form (53), be applied to S_1', we get for the coordinates, with respect to fixed rectangular axes, of the new surfaces

$$\xi = e^{-\alpha_1}\left(\frac{\partial x_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial x_1'}{\partial \gamma_1}\right), \quad \eta = e^{-\alpha_1}\left(\frac{\partial y_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial y_1'}{\partial \gamma_1}\right), \quad \zeta = e^{-\alpha_1}\left(\frac{\partial z_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial z_1'}{\partial \gamma_1}\right).$$

But the coordinates of S_1' with reference to these axes are

$$x_1' = x + \cos\theta_1 X_1 + \sin\theta_1 X_2,$$

and analogous expressions for y_1' and z_1'. When these values are substituted in the above and one takes account of (46), (50) and (55), one gets

$$\xi = e^{-\alpha_1}(\cos\theta_1 X_1 + \sin\theta_1 X_2) + \gamma_1 X, \tag{56}$$

and similarly for η and ζ. From this we have by differentiation

$$\left.\begin{aligned} \frac{\partial \xi}{\partial u} &= -(e^{-\alpha_1}\cos\omega - \gamma_1 \sin\omega)\, X_1, \\ \frac{\partial \xi}{\partial v} &= -(e^{-\alpha_1}\sin\omega + \gamma_1\cos\omega)\, X_2, \end{aligned}\right\} \tag{57}$$

from which it is seen that the tangent planes to Σ and S are parallel. Moreover, these equations show that the parametric system on Σ is orthogonal, and being represented on the sphere by an orthogonal system, they are the lines of curvature for Σ, which shows that the latter is an A-surface. From (57) we have

$$A' = (e^{-\alpha_1}\cos\omega - \gamma_1\sin\omega), \quad C' = -(e^{-\alpha_1}\sin\omega + \gamma_1\cos\omega); \tag{58}$$

it is readily found that these values satisfy the conditions (6). From (56) we find

$$2q = e^{-2\alpha_1} + \gamma_1^2, \quad p = \gamma_1,$$

and from (58),

$$\rho_1 = \gamma_1 - e^{-\alpha_1}\cot\omega, \quad \rho_2 = \gamma_1 + e^{-\alpha_1}\tan\omega.$$

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* C. R. March 23, 1891, and March 13, 1893.

When these values are substituted in (53), it is found that k is zero, so that the surface defined by (56) is a surface of Bianchi of the parabolic type. Moreover, each particular integral of equations (38) gives a new surface of Bianchi of this type.

We pass now to the determination of the surfaces of Bonnet, which are the transforms of the above surface Σ. From (58) and (42) we have for the corresponding values of λ and μ,

$$\lambda = -e^{-a_1}, \quad \mu = -\gamma_1,$$

so that the coordinates of the transform are

$$\xi_1 = \xi - e^{-a_1}(\cos\theta X_1 + \sin\theta X_2) - \gamma_1 X,$$

and similarly for η_1 and ζ_1, where X_1, Y_1, Z_1; X_2, Y_2, Z_2 are the direction-cosines of the tangents to the lines of curvature of Σ, and since these are parallel to the corresponding tangents for S, they have the values given by (54) and (55). When the value of ξ from (56) is substituted in the above, the latter becomes

$$\xi_1 = e^{-a_1}[(\cos\theta_1 - \cos\theta)X_1 + (\sin\theta_1 - \sin\theta)X_2]. \tag{59}$$

From this and similar expressions for η_1 and ζ_1, it is evident that when Σ is transformed by means of the same angle θ_1 which is used in its definition, the transform Σ_1 reduces to a point, namely, the origin in the above system of coordinates. For the other transforms we have from (59),

$$\left.\begin{aligned}\frac{\partial\xi_1}{\partial u} &= e^{-a_1}(\cos\theta_1 - \cos\theta)[X_1\cos\omega\cos\theta + X_2\cos\omega\sin\theta - X\sin\omega],\\ \frac{\partial\xi_1}{\partial v} &= e^{-a_1}(\sin\theta_1 - \sin\theta)[X_1\sin\omega\cos\theta + X_2\sin\omega\sin\theta + X\cos\omega].\end{aligned}\right\} \tag{60}$$

From these we get for the coefficients of the linear element of Σ_1,

$$A_1 = e^{-a_1}(\cos\theta_1 - \cos\theta), \quad C_1 = e^{-a_1}(\sin\theta_1 - \sin\theta). \tag{61}$$

From (59) we have

$$\Sigma\xi_1 X = 0,$$

that is, the radii vectores of the surfaces Σ_1 are parallel to the corresponding tangent planes to Σ.

The preceding investigations show that the normals to Σ_1 have for direction-cosines the expressions

$$\sin\theta X_1 - \cos\theta X_2, \quad \sin\theta Y_1 - \cos\theta Y_2, \quad \sin\theta Z_1 - \cos\theta Z_2,$$

so that from (59) we have, for the distances from the origin to the tangent planes to Σ_1 and half the square of the distance to the point of contact, the expression

$$p_1 = -e^{-a_1}\sin(\theta_1 - \theta), \quad q_1 = e^{-2a_1}[1 - \cos(\theta_1 - \theta)].$$

Again, the principal radii of curvature have the expressions

$$\rho_1 = \frac{e^{-a_1}(\cos\theta_1 - \cos\theta)}{\sin\theta}, \quad \rho_2 = \frac{e^{-a_1}(\sin\theta - \sin\theta_1)}{\sin\theta}.$$

When these values are substituted in (53), the latter is satisfied identically if k is zero. Hence:

All the transforms of a surface of Bianchi of the parabolic type are surfaces of Bianchi of the same type and one reduces to a point. From the previous remarks about the associated cyclic systems, it follows that all the circles must pass through this point, so that—

The circles of the cyclic system whose axes are the normals to a surface of Bianchi of the parabolic type, pass through a point and the infinity of surfaces orthogonal to these circles are surfaces of Bianchi of the same type.

We have seen that the transforms of these surfaces Σ_1 by the angle $\omega + \pi$ are one surface, namely, the envelope of the plane parallel to the tangent plane to Σ and at a distance μ from it. Moreover, this is also the envelope of the planes of the circles of the cyclic system associated with Σ. The function λ_1 for all the transformations of the surfaces Σ_1 is found from (61) by (42) to have the value

$$\lambda_1 = e^{-a_1}[\cos(\theta - \theta_1) - 1].$$

When this value and those for A_1 and C_1, given by (61), are substituted in formulæ similar to (44) and $\omega + \pi$ is the angle of transformation, we find that the coefficients A'', C'' of the linear element of this transform take the value zero,

so that the transform is a point. We know that the coordinates of this point with respect to the trihedron of Σ are

$$\lambda \cos\theta, \quad \lambda \sin\theta, \quad \mu,$$

where λ and θ are given by (51′) and μ is $-\gamma_1$. If the values for A and C from (58) are substituted in (51′), it is found that

$$\lambda = -e^{-\alpha_1}, \quad \theta = \theta_1 .$$

This shows that the original circle for Σ passes through this point, so that we have the following theorem:

The surface of the same class as Σ, which is the transform of all the surfaces Σ_1, is a point and the same one through which pass all the circles of the cyclic system associated with Σ.

But every solution of the system (38) gives a surface Σ, and the above point is the origin in every case. Hence the theorem:

Among all the normal cyclic congruences, whose developables have the same spherical representation, there is an infinity whose associated circles pass through a point, the same for all the congruences.

Bianchi has shown that this theorem is true for all cyclic congruences.*

In a purely geometrical manner, Bianchi has established the following theorem concerning surfaces Σ: †

If one take any surface Σ_1 of the parabolic type and a sphere S_0 with center at the origin, the circles normal to Σ_1 and which cut the sphere in diametrically opposite points or orthogonally admit an infinity of orthogonal surfaces which are in every case surfaces Σ of the parabolic type, and the axes of these circles are cut normally by a family of surfaces Σ of the elliptic type in the former case and of the hyperbolic type in the latter; when S_0 reduces to a point, the latter family is composed of surfaces of the parabolic type.

In accordance with this theorem, Bianchi finds the analytical expressions for these surfaces. We shall recall his results briefly and then apply to these surfaces the above transformations.

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* Lezioni, p. 335; Ger. trans., p. 353. † Annali, l. c., p. 367.

Bianchi takes for the surface Σ_1 of the parabolic type, one which is derived from the pseudospherical surface S in a manner similar to the derivation of Σ from S_1''. The coordinates of a point on Σ_1 are found to have the expressions

$$\xi_1 = (e^a \cos\theta_1 + \beta \sin\theta_1) X_1 + (e^a \sin\theta_1 - \beta \cos\theta_1) X_2,$$

and, similarly, for η_1 and ζ_1; α, β are the functions given by (46) and $X_1 \ldots Z_2$ by (54).

We denote by ξ_0, η_0, ζ_0 the coordinates of the center and by R the radius of the circle, drawn as indicated in the above theorem, and so that the plane of the circle be parallel to the tangent plane to S; we have

$$\xi_0 = \xi_1 + R(\cos\theta_1 X_1 + \sin\theta_1 X_2),$$

whence

$$\xi_0 = [(e^a + R)\cos\theta_1 + \beta \sin\theta_1] X_1 + [(e^a + R)\sin\theta_1 - \beta\cos\theta_1] X_2,$$

and, similarly, for η_0 and ζ_0. In accordance with the above definition of the surfaces, one must have

$$\Sigma \xi_0^2 = R^2 - k,$$

where k is a constant, positive or negative according as the circles cut the sphere of radius $\sqrt{\pm k}$ in diametrically opposite points or orthogonally. This condition leads to

$$e^a + R = \tfrac{1}{2}[e^a - (\beta^2 + k) e^{-a}],$$

so that the above expression for ξ_0 can be written

$$\xi_0 = [\tfrac{1}{2}\{e^a - (\beta^2 + k)e^{-a}\}\cos\theta_1 + \beta\sin\theta_1] X_1 \\ + [\tfrac{1}{2}\{e^a - (\beta^2 + k)e^{-a}\}\sin\theta_1 - \beta\cos\theta_1] X_2. \quad (62)$$

The congruence of the axes of these circles is defined by the formula

$$\xi = \xi_0 + tX, \quad \eta = \eta_0 + tY, \quad \zeta = \zeta_0 + tZ.$$

For these formulæ to define a surface normal to the congruence, t must satisfy the condition

$$\Sigma X d\xi = 0;$$

this gives for the determination of t, in consequence of (62), the equation

$$dt = [\tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\cos\theta_1 + \beta\sin\theta_1]\sin\omega du$$
$$- [\tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\sin\theta_1 - \beta\cos\theta_1]\cos\omega dv; \quad (63)$$

the right-hand member can readily be shown, by means of (38) and (46), to be an exact differential. Hence, the normal surfaces are given by

$$\xi = [\tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\cos\theta_1 + \beta\sin\theta_1]X_1$$
$$+ [\tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\sin\theta_1 - \beta\cos\theta_1]X_2 + tX, \quad (64)$$

and analogous expressions for η and ζ. From (64) one gets

$$\left.\begin{aligned} \frac{\partial\xi}{\partial u} &= [\tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\cos\omega + t\sin\omega]X_1, \\ \frac{\partial\xi}{\partial v} &= [\tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\sin\omega - t\cos\omega]X_2, \end{aligned}\right\} \quad (65)$$

from which it is readily seen that the parametric system on these surfaces is orthogonal, and since these lines have the spherical representation (4), it follows that the surfaces defined by (64) are surfaces of Bonnet. From (65) we have

$$\left.\begin{aligned} A &= \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\cos\omega + t\sin\omega, \\ C &= \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\sin\omega - t\cos\omega, \end{aligned}\right\} \quad (66)$$

which values are readily found to satisfy the condition (6). Bianchi* shows that the equation (53) is satisfied by these surfaces and k has the same meaning in both. We proceed now to find the transforms of these surfaces.

In accordance with the formulæ (42), one has at once

$$\lambda = \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}, \quad \mu = -t, \quad (67)$$

and by (44),

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\cos\theta_1 + \beta\sin\theta_1 + \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\cos\theta, \\ C_1 &= \tfrac{1}{2}\{e^{a} - (\beta^2 + k)e^{-a}\}\sin\theta_1 - \beta\cos\theta_1 + \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}\sin\theta. \end{aligned}\right\} \quad (68)$$

We can find directly the coordinates ξ_1, η_1, ζ_1, with reference to fixed axes, of these surfaces. By means of (67), we have

$$\xi_1 = \xi + \tfrac{1}{2}\{e^{a} + (\beta^2 + k)e^{-a}\}[X_1\cos\theta + X_2\sin\theta] - tX,$$

* Annali, l. c., p. 368.

and analogous expressions for η_1 and ζ_1. If the value for ξ from (64) be put in the right-hand member, this becomes

$$\xi_1 = [\tfrac{1}{2} e^{a} (\cos\theta_1 + \cos\theta) + \tfrac{1}{2} (\beta^2 + k) e^{-a} (\cos\theta - \cos\theta_1) + \beta \sin\theta_1] X_1$$
$$+ [\tfrac{1}{2} e^{a} (\sin\theta_1 + \sin\theta) + \tfrac{1}{2} (\beta^2 + k) e^{-a} (\sin\theta - \sin\theta_1) - \beta\cos\theta_1] X_2. \quad (69)$$

When in particular, Σ is transformed by the angle θ_1, the transform is given by

$$\xi_1 = [e^{a} \cos\theta_1 + \beta\sin\theta_1] X_1 + [e^{a}\sin\theta_1 - \beta\cos\theta_1] X_2,$$

and similar expressions in η_1 and ζ_1. But this is the surface Σ_1 from which we started, and hence is of the parabolic type. Moreover, from the defining theorem of Bianchi, it follows that *all the surfaces defined by* (69) *are surfaces* Σ *of the parabolic type*, for they form the infinite family of surfaces which cut orthogonally the circles whose axes are normal to the surfaces Σ defined by (64). When k is zero, one of these surfaces must reduce to a point, namely, the origin. To find the value of θ, corresponding we have only to equate A_1 and C_1 to zero, which gives

$$(e^{a} + \beta^2 e^{-a}) \cos\theta = (\beta^2 e^{-a} - e^{a}) \cos\theta_1 - 2\beta \sin\theta_1,$$
$$(e^{a} + \beta^2 e^{-a}) \sin\theta = (\beta^2 e^{-a} - e^{a}) \sin\theta_1 + 2\beta\cos\theta_1,$$

and when these values are put in (69), we get the desired result.

Again, since the surfaces Σ_1 are of the parabolic type, the circles of their associate cyclic system must pass through a point. To find this point, we have only to determine the value of ϕ which makes Σ_2 reduce to a point. From (68) we have, for λ_1 and μ_1,

$$\lambda_1 = \tfrac{1}{2}\{e^{a} + (\beta^2 + k) e^{-a}\} + \tfrac{1}{2}\{e^{a} - (\beta^2 + k) e^{-a}\} \cos(\theta_1 - \theta) + \beta\sin(\theta_1 - \theta),$$
$$\mu_1 = \tfrac{1}{2}\{e^{a} - (\beta^2 + k) e^{-a}\} \sin(\theta_1 - \theta) - \beta\cos(\theta_1 - \theta),$$

so that if we determine A_2 and C_2 from expressions similar to (44), we get

$$A_2 = \lambda_1(\cos\omega + \cos\phi), \quad C_2 = \lambda_1(\sin\omega + \sin\phi).$$

From this it is seen that Σ_2 is a point, when any of the surfaces defined by (69) are transformed by the angle $\omega + \pi$. On this account the plane through this point is parallel to the tangent plane to Σ and, consequently, the length λ_1 is

measured along the line whose direction-cosines are

$$-(X_1 \cos\theta + X_2 \sin\theta), \quad -(Y_1 \cos\theta + Y_2 \sin\theta), \quad -(Z_1 \cos\theta + Z_2 \sin\theta),$$

and μ_1 along the line whose direction-cosines are

$$X_1 \sin\theta - X_2 \cos\theta, \quad Y_1 \sin\theta - Y_2 \cos\theta, \quad Z_1 \sin\theta - Z_2 \cos\theta. \qquad (70)$$

Hence the coordinates of this particular surface Σ_2 are

$$\xi_2 = \xi_1 - \lambda_1 (X_1 \cos\theta + X_2 \sin\theta) + \mu_1 (X_1 \sin\theta - X_2 \cos\theta),$$

and similar ones for η_2 and ζ_2. When the values for ξ_1, η_1, ζ_1 are substituted from (69), they reduce to zero, so that Σ_2 is the origin. Moreover, this is entirely independent of the angle θ by which the surfaces Σ were transformed. Hence:

The circles of the cyclic system associated with the transform of a surface of Bianchi (64) *pass through the same point as the circles of the cyclic system associated with a surface Σ of the parabolic type, with the same spherical representation of its lines of curvature as the original surface.*

From (68), (69) and (70), we get, for the quantities p_1, q_1, ρ_1 and ρ_2 for the surface Σ_1 the values

$$\begin{aligned} 2q_1 = \tfrac{1}{2}\,[e^{2\alpha} + e^{-2\alpha}(\beta^2 + k)^2] + \tfrac{1}{2}\,[e^{2\alpha} - e^{-2\alpha}(\beta^2 + k)^2] \cos(\theta_1 - \theta) + \beta^2 \\ + \beta\,[e^{\alpha} + e^{-\alpha}(\beta^2 + k)] \sin(\theta_1 - \theta), \end{aligned}$$

$$p_1 = -\tfrac{1}{2}\,[e^{\alpha} - e^{-\alpha}(\beta^2 + k)] \sin(\theta_1 - \theta) + \beta \cos(\theta_1 - \theta),$$

$$\rho_1 = \frac{A_1}{\sin\theta}, \quad \rho_2 = -\frac{C_1}{\cos\theta}.$$

Between these functions there obtains the relation

$$2q_1 - p_1(\rho_1 + \rho_2) + \rho_1\rho_2 = 0,$$

which shows that the surfaces Σ_1 are parabolic and that the associated circles pass through the origin, affording a verification of the preceding.

§7.—*General Transformations of A-surfaces.*

We proceed now to the definition of more general transformations of the A-surfaces. We consider the envelope of the plane which cuts the tangent plane to such a surface under a constant angle σ and along the line through the point of tangency which makes an angle θ with the tangent to the line of curvature $v=$ const. through the point. It will be found that for determinate values of θ this envelope is an A-surface.

As before we consider S with reference to the moving trihedron; denote by M the point of tangency on S and M_1 the corresponding point on the envelope S_1. In the enveloping plane we drop from M_1 a perpendicular upon the line of intersection of this plane with the tangent plane to S, and denote its length, by μ. Further, we let λ denote the length from M to the foot of this perpendicular. Then the coordinates of M_1 with respect to the moving axes have the expressions

$$x=\lambda\cos\theta-\mu\cos\sigma\sin\theta,\quad y=\lambda\sin\theta+\mu\cos\sigma\cos\theta,\quad z=\mu\sin\sigma. \tag{80}$$

The projections upon the axes of a displacement of M_1 have the values* from (5)

$$dx+(A+\mu\sin\omega\sin\sigma)\,du-(\lambda\sin\theta+\mu\cos\sigma\cos\theta)\left(\frac{\partial\omega}{\partial v}\,du+\frac{\partial\omega}{\partial u}\,dv\right),$$

$$dy+(C-\mu\cos\omega\sin\sigma)\,dv+(\lambda\cos\theta-\mu\cos\sigma\sin\theta)\left(\frac{\partial\omega}{\partial v}\,du+\frac{\partial\omega}{\partial u}\,dv\right).$$

$$dz+\cos\omega\,(\lambda\sin\theta+\mu\cos\sigma\cos\theta)\,dv-\sin\omega\,(\lambda\cos\theta-\mu\cos\sigma\sin\theta)\,du.$$

We project these three lengths upon the three orthogonal directions formed by the line, MP, of the intersection of the two planes, the line, MQ, in the tangent plane perpendicular to this and the normal, MN, to the surface. These projections are readily found to be

$$\left.\begin{aligned}
&d\lambda-\mu\cos\sigma d\theta+A\cos\theta du+C\sin\theta dv-\mu\cos\sigma\left(\frac{\partial\omega}{\partial v}\,du+\frac{\partial\omega}{\partial u}dv\right)\\
&\qquad\qquad+\mu\sin\sigma\,(\sin\omega\cos\theta\,du-\cos\omega\sin\theta dv),\\
&\lambda d\theta+\cos\sigma d\mu-A\sin\theta du+C\cos\theta dv+\lambda\left(\frac{\partial\omega}{\partial v}\,du+\frac{\partial\omega}{\partial u}\,dv\right)\\
&\qquad\qquad-\mu\sin\sigma\,(\sin\omega\sin\theta du+\cos\omega\cos\theta dv),\\
&\sin\sigma d\mu+\cos\omega\,(\lambda\sin\theta+\mu\cos\sigma\cos\theta)-\sin\omega\,(\lambda\cos\theta-\mu\cos\sigma\sin\theta).
\end{aligned}\right\}\tag{81}$$

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* Darboux, Leçons, Vol. 2, p. 385.

The direction-cosines of the given plane with respect to the lines MP, MQ, MN are evidently

$$0, \; -\sin\sigma_1, \; \cos\sigma.$$

Consequently, the necessary and sufficient condition that this plane envelope S_1 is that these functions satisfy the equation

$$\begin{aligned}\left(\lambda\sin\sigma\frac{\partial\theta}{\partial u} - A\sin\theta\sin\sigma + \lambda\frac{\partial\omega}{\partial v}\sin\sigma + \lambda\cos\theta\sin\omega\cos\sigma - \mu\sin\omega\sin\theta\right)du \\ + \left(\lambda\sin\sigma\frac{\partial\theta}{\partial v} + C\sin\sigma\cos\theta + \lambda\frac{\partial\omega}{\partial u}\sin\sigma - \lambda\sin\theta\cos\omega\cos\sigma \right. \\ \left. -\mu\cos\omega\cos\theta\right)dv = 0. \qquad (82)\end{aligned}$$

We consider first the case where S is pseudospherical, so that we can take

$$A = \cos\omega, \quad C = \sin\omega. \qquad (83)$$

When these values are substituted in (82), we get, by equating to zero, the coefficients of du and dv, the following equations of condition:

$$\left.\begin{aligned}\lambda\sin\sigma\left(\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v}\right) &= \sin\sigma\sin\theta\cos\omega - \lambda\cos\sigma\cos\theta\sin\omega + \mu\sin\theta\sin\omega, \\ \lambda\sin\sigma\left(\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= -\sin\sigma\cos\theta\sin\omega + \lambda\cos\sigma\sin\theta\cos\omega + \mu\cos\theta\cos\omega.\end{aligned}\right\} (84)$$

Taking the derivative of the first with respect to v and the second with respect to u, subtracting and noting that ω satisfies equation (7), we have

$$\begin{aligned}(\sin\sigma\sin\theta\cos\omega + \mu\sin\theta\sin\omega)\frac{\partial\log\lambda}{\partial v} \\ + (\sin\sigma\cos\theta\sin\omega - \mu\cos\theta\cos\omega)\frac{\partial\log\lambda}{\partial u} \\ + \frac{(\sin^2\sigma - \lambda^2 - \mu^2)\sin\omega\cos\omega}{\lambda\sin\sigma} - \sin\theta\sin\omega\frac{\partial\mu}{\partial v} + \cos\theta\cos\omega\frac{\partial\mu}{\partial u} = 0. \qquad (85)\end{aligned}$$

If, in turn, we take the derivative of the first of (84) with respect to u and the second with respect to v and subtract and *then make the hypothesis that θ is a solu-*

tion of equation (7), we are brought to the equation

$$(\sin\sigma\cos\theta\sin\omega - \mu\cos\theta\cos\omega)\frac{\partial\log\lambda}{\partial v}$$
$$+ (\sin\sigma\sin\theta\cos\omega + \mu\sin\theta\sin\omega)\frac{\partial\log\lambda}{\partial u}$$
$$+\frac{(\lambda^2-\mu^2-\sin^2\sigma)\sin\theta\cos\theta+\lambda\mu\cos\sigma(\cos^2\theta-\sin^2\theta)}{\lambda\sin\sigma}$$
$$+\cos\theta\cos\omega\frac{\partial\mu}{\partial v} - \sin\theta\sin\omega\frac{\partial\mu}{\partial u} = 0. \tag{86}$$

If, now, we have given two functions λ and μ satisfying equations (85) and (86) and these values be substituted in (84), the latter will define a function θ_1 involving an arbitrary constant, which satisfies equation (7) and such that the surface defined by (80) is the envelope of this plane of angle σ with the tangent plane to S.

Consider the case where λ and μ are constant. Equations (85) and (86) reduce to

$$(\sin^2\sigma - \lambda^2 - \mu^2)\sin\omega\cos\omega = 0,$$
$$(\lambda^2-\mu^2-\sin^2\sigma)\sin\theta\cos\theta + \lambda\mu\cos\sigma(\cos^2\theta - \sin^2\theta) = 0.$$

Since λ and μ are independent of θ, these may be replaced by

$$\sin^2\sigma - \lambda^2 - \mu^2 = 0, \quad \lambda^2 - \mu^2 - \sin^2\sigma = 0, \quad \lambda\mu = 0,$$

from which it follows that

$$\lambda^2 = \sin^2\sigma, \quad \mu = 0. \tag{87}$$

If we take

$$\lambda = \sin\sigma, \quad \mu = 0, \tag{87'}$$

the equations (84) become

$$\left.\begin{aligned}\sin\sigma\left(\frac{\partial\theta}{\partial u}+\frac{\partial\omega}{\partial v}\right) &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega,\\ \sin\sigma\left(\frac{\partial\theta}{\partial v}+\frac{\partial\omega}{\partial u}\right) &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega.\end{aligned}\right\} \tag{88}$$

When the values from (87′) and (83) are substituted in (81) and in the reduction use is made of equations (88), one gets for the projections of the dis-

placement of M_1,

$$\left.\begin{aligned}&\cos\omega\cos\theta du+\sin\omega\sin\theta dv,\\ -&\cos\sigma(\sin\omega\cos\theta du-\cos\omega\sin\theta dv),\\ -&\sin\sigma(\sin\omega\cos\theta du-\cos\omega\sin\theta dv),\end{aligned}\right\}\tag{89}$$

and from these we get for the linear element of S_1 the expression

$$ds_1^2=\cos^2\theta du^2+\sin^2\theta dv^2.\tag{89'}$$

These are the same results which Darboux gets in his study of the transformations of pseudospherical surfaces.* From them he shows that the parametric curves on S_1 are its lines of curvature so that S_1 is the pseudospherical surface corresponding to the solution θ of equation (7). This transformation, which is equivalent to finding the locus of points on the lines tangent to the curves on S_1 given by a solution of the system (88), and at a distance $\sin\sigma$, was discovered by Bäcklund and has since that time been called by his name. It is evident that the transformations which we are discussing are a generalization of these transformations of Bäcklund.

The other solution, $\lambda=-\sin\sigma$ of (87) offers nothing new as it merely gives a Bäcklund transformation of different angles.

We return now to the case of consideration of A-surfaces in general and discuss the case where θ satisfies equations (88). For this choice of θ, the equation (82) can be replaced by

$$\left.\begin{aligned}A\sin\sigma&=\lambda\cos\omega-\mu\sin\omega,\\ C\sin\sigma&=\lambda\sin\omega+\mu\cos\omega,\end{aligned}\right\}\tag{90}$$

from which it follows that

$$\left.\begin{aligned}\lambda&=(A\cos\omega+C\sin\omega)\sin\sigma,\\ \mu&=(-A\sin\omega+C\cos\omega)\sin\sigma.\end{aligned}\right\}\tag{91}$$

Substituting these values in (81), these projections of the displacement of M_1 become

$$\left.\begin{aligned}&A_1\cos\omega\, du+C_1\sin\omega\, dv,\\ -&\cos\sigma\,(A_1\sin\omega\, du-C_1\cos\omega\, dv),\\ -&\sin\sigma\,(A_1\sin\omega\, du-C_1\cos\omega\, dv),\end{aligned}\right\}\tag{92}$$

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* Darboux, Leçons, Vol. 3, p. 435.

where we have put for brevity

$$\left.\begin{aligned} A_1 &= \sin\sigma\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + \lambda\frac{\cos\theta}{\sin\sigma} - \mu\frac{\cos\sigma\sin\theta}{\sin\sigma}, \\ C_1 &= \sin\sigma\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) + \lambda\frac{\sin\theta}{\sin\sigma} + \mu\frac{\cos\sigma\cos\theta}{\sin\sigma}. \end{aligned}\right\} \tag{93}$$

From this we find for the linear element of the envelope S_1

$$ds_1^2 = A_1^2 du^2 + C_1^2 dv^2. \tag{94}$$

By a system of reasoning similar to that followed in the case where σ was $\pi/2$ we can show that the lines $u = \text{const.}$, $v = \text{const.}$ are lines of curvature and hence S_1 is an A-surface. Hence the theorem:

Given an A-surface S, corresponding to a solution ω of equation (7), *and given also a solution θ of equations* (88) *for any angle σ. The plane which cuts the tangent plane to S under the angle σ and along the line through the point of contact which makes the angle θ with the tangent to the line of curvature $v = \text{const.}$ is an A-surface with the spherical representation*

$$\sin^2\theta du^2 + \cos^2\theta dv^2. \tag{95}$$

When $\sigma = \pi/2$ we call the transformation *complementary*.

As in the case where σ was a right angle, the expressions for μ and λ are independent of θ, so that the points of tangency with their envelopes of all planes through a given point M of S lie on circles, each angle σ giving rise to a separate circle, and the planes of these circles are parallel to the tangent plane to S and approach it as σ approaches zero. Hence with every A-surface there is associated not only a cyclic system of circles, but an infinity of systems of circles, the circles of each system being cut under the same angle by a family of A-surfaces. But, as we have seen, these are the only cyclic systems whose circles have axes forming a normal congruence; consequently, we have the theorem:

The lines of a normal cyclic congruence are the axes of an infinity of circles which are cut under different constant angles by families of surfaces.

From the definition of μ and its expression (91) it is evident that the planes of the circles of these systems are at the distance

$$(-A \sin \omega + C \cos \omega) \sin^2 \sigma$$

from the tangent plane to S, which shows that the distance decreases with σ.

We have found (20) that the linear element of a surface parallel to a pseudospherical surface has the linear element

$$ds^2 = (\cos \omega - a \sin \omega)^2 du^2 + (\sin \omega + a \cos \omega) dv^2.$$

If these values for A and C be substituted in (91), we get

$$\lambda = \sin \sigma, \qquad \mu = a \sin \sigma.$$

When all these values are substituted in (93), the latter become

$$A_1 = \cos \theta - a \cos \sigma \sin \theta_1 \qquad C_1 = \sin \theta + a \cos \sigma \cos \theta.$$

A comparison of these expressions with the coefficients of the above linear element makes it clear that the transform of a parallel, S, of a pseudospherical surface Σ is parallel to the transform Σ by means of the same angles θ and σ. From the values of A_1 and C_1 it follows that the transform of S is pseudospherical only when σ is $\pi/2$.

As in the case of transformations with σ a right angle, it is seen that when S_1 is found the complete determination of the other A-surfaces with the same spherical representation of their lines of curvature requires the integration of (47), where θ has the value given by the solution of equations (88).

If, with Bianchi,* we denote by B_σ the Bäcklund transformation of angle σ, it can be shown, as Lie first pointed out, that

$$B_\sigma = L_\sigma B_{\frac{\pi}{2}} L_\sigma^{-1},$$

where L_σ is given by (29). This result, which was established for transformation of pseudospherical surfaces, is just as valid for transformation of any A-surface. for in its derivation no use is made of the linear element of the surface.

* Bianchi, Lezioni, p. 434; Ger. trans., 460.

Hence, all the transformations of surfaces which we have been discussing can be obtained by transformations of Lie and complementary transformations.

§8.—*Theorem of Permutability for A-surfaces.*

Bianchi has established for the transformations of Bäcklund the *theorem of permutability* by means of which it can be shown that when one can find all the transforms S_1 of a given pseudospherical surface S, the transformations of these surfaces S_1 are given by algebraic processes and differentiation. This theorem is readily established when the surface is referred to the moving trihedron. After deducing it, we shall apply it to the theory of transformations of A-surfaces.

Let S_1 be the transform of S corresponding to the angles θ_1 and σ_1, and S_2 for the angles θ_2, σ_2. Let ϕ be a solution of the system

$$\left.\begin{aligned}\sin\sigma_2\left(\frac{\partial\phi}{\partial u}+\frac{\partial\theta_1}{\partial v}\right) &= \sin\phi\cos\theta_1-\cos\sigma_2\cos\phi\sin\theta_1,\\ \sin\sigma_2\left(\frac{\partial\phi}{\partial v}+\frac{\partial\theta_1}{\partial u}\right) &= -\cos\phi\sin\theta_1+\cos\sigma_2\sin\phi\cos\theta_1,\end{aligned}\right\}\tag{96}$$

and denote by S_3 the corresponding transform of S_1. It is evident that the projections on the tangents to the lines of curvature and the normal to S_1 of the line $M_1 M_3$ are

$$\sin\sigma_2\cos\phi,\quad \sin\sigma_2\sin\phi,\quad 0.$$

Again, if we denote by X_1', Y_1', Z_1'; X_1'', Y_1'', Z_1'', the direction-cosines of these tangents to the lines of curvature of S_1 with respect to the directions MP, MQ, MR for S, we find from (89) and (89′),

$$\left.\begin{aligned}X_1',\ Y_1',\ Z_1' &= \cos\omega,\ -\cos\sigma_1\sin\omega,\ -\sin\sigma_1\sin\omega,\\ X_2',\ Y_2',\ Z_2' &= \sin\omega,\quad \cos\sigma_1\cos\omega,\quad \sin\sigma_1\cos\omega.\end{aligned}\right\}\tag{98}$$

From these expressions and (97) it follows that the coordinates of M_3, with respect to the axes MP, MQ, MN, have the expressions

$$\xi=\sin\sigma_1+\sin\sigma_2\cos(\phi-\omega),\quad \eta=\sin\sigma_2\cos\sigma_1\sin(\phi-\omega),$$
$$\zeta=\sin\sigma_1\sin\sigma_2\sin(\phi-\omega).$$

Recalling the definition of MP and MQ, we find for the coordinates, x_3, y_3, z_3, of M_3 with respect to the axes of the moving trihedron

$$\left.\begin{aligned}x_3 &= \cos\theta_1 \sin\sigma_1 + \cos\theta_1 \sin\sigma_2 \cos(\phi-\omega) - \sin\theta_1 \sin\sigma_2 \cos\sigma_1 \sin(\phi-\omega),\\ y_3 &= \sin\theta_1 \sin\sigma_1 + \sin\theta_1 \sin\sigma_2 \cos(\phi-\omega) + \cos\theta_1 \sin\sigma_2 \cos\sigma_1 \sin(\phi-\omega),\\ z_3 &= \sin\sigma_1 \sin\sigma_2 \sin(\phi-\omega).\end{aligned}\right\} \quad (99)$$

We assume that there is a solution ϕ of equations (96) which satisfies the equations

$$\left.\begin{aligned}\sin\sigma_1\left(\frac{\partial\phi}{\partial u}+\frac{\partial\theta_2}{\partial v}\right) &= \sin\phi\cos\theta_2 - \cos\sigma_1\cos\phi\sin\theta_2,\\ \sin\sigma_1\left(\frac{\partial\phi}{\partial v}+\frac{\partial\theta_2}{\partial u}\right) &= -\cos\phi\sin\theta_2 + \cos\sigma_1\sin\phi\cos\theta_2.\end{aligned}\right\} \quad (100)$$

If S_2 is transformed by means of this same function ϕ and the angle σ_1, we find for the coordinates, denoted by x'_3, y'_3, z'_3, of the transform expressions which can be obtained from (99) by interchanging the subscripts 1 and 2. Then z_3 and z'_3 are equal; we assume further that x_3 and x'_3 are equal, and also y_3 and y'_3. It is necessary that we have

$$\begin{aligned}\cos\theta_1(x_3-x'_3)+\sin\theta_1(y_3-y'_3)&=0,\\ \cos\theta_2(x_3-x'_3)+\sin\theta_2(y_3-y'_3)&=0;\end{aligned}$$

and these equations are the sufficient condition for the equality of the coordinates, for otherwise $\sin(\theta_1-\theta_2)$ is zero which is evidently impossible.

If the above expressions for x_3, y_3, x'_3, y'_3 are substituted in these equations they become

$$\left.\begin{aligned}&[\sin\sigma_1\cos(\theta_2-\theta_1)-\sin\sigma_2]\cos(\phi-\omega)\\ &\quad -\sin\sigma_1\cos\sigma_2\sin(\theta_2-\theta_1)\sin(\phi-\omega)=\sin\sigma_1-\sin\sigma_2\cos(\theta_2-\theta_1),\\ &[\sin\sigma_2\cos(\theta_2-\theta_1)-\sin\sigma_1]\cos(\phi-\omega)\\ &\quad -\sin\sigma_2\cos\sigma_1\sin(\theta_2-\theta_1)\sin(\phi-\omega)=\sin\sigma_2-\sin\sigma_1\cos(\theta_2-\theta_1).\end{aligned}\right\} \quad (101)$$

Solving these equations with respect to $\sin(\phi-\omega)$ and $\cos(\phi-\omega)$, one gets

$$\left.\begin{aligned}\sin(\phi-\omega)&=\frac{(\cos\sigma_2-\cos\sigma_1)\sin(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1},\\ \cos(\phi-\omega)&=\frac{\sin\sigma_1\sin\sigma_2+(\cos\sigma_1\cos\sigma_2-1)\cos(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1},\end{aligned}\right\} \quad (102)$$

which are readily found to satisfy the condition that the sum of the squares is equal to unity. From these we get

$$\tan\left(\frac{\phi-\omega}{2}\right)=\frac{\sin\left(\frac{\sigma_1+\sigma_2}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma_2}{2}\right)}\tan\left(\frac{\theta_1-\theta_2}{2}\right), \tag{103}$$

which is similar to the result of Bianchi*. As he has remarked, it is readily shown that this value of ϕ satisfies equations (96) and (100), so that the preceding hypotheses are consistent. Hence the transform of S_1 by ϕ and the angle σ_2 is the same pseudospherical surface as the transform of S_2 by ϕ and the angle σ_1, which is the so-called *theorem of permutability*.

Suppose then that one has the general solution of the system (88), say

$$\theta\,(u,\,v,\,\sigma,\,c)\,.$$

Let S_1' denote the transform of S corresponding to the function $\theta\,(u,\,v,\,\sigma_1,\,c_1)$; then all the transforms of S_1' are obtained by means of the function ϕ given by

$$\tan\frac{\phi-\omega}{2}=\frac{\sin\left(\frac{\sigma+\sigma_1}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma}{2}\right)}\tan\frac{\theta_1-\theta}{2}, \tag{104}$$

for all values of σ except σ_1. For this exceptional case, the right-hand side is indeterminate, but Bianchi shows† that, if one substitute in place of $\tan\frac{\theta_1-\theta}{2}\Big/\sin\frac{\sigma_1-\sigma}{2}$, which assumes the form $0/0$, the quotient of the derivatives with respect to σ, the formula becomes

$$\tan\frac{\phi-\omega}{2}=\sin\sigma_1\left[\frac{\partial\theta}{\partial\sigma}+c'\frac{\partial\theta}{\partial c}\right]_{\sigma=\sigma_1}, \tag{105}$$

where c' is a new constant; and then it is shown that this function ϕ satisfies the condition of the problem. In considering the case where $\sigma=\sigma_1$, we mean a

* Lezioni, p. 487; Ger. trans., p. 464. The difference between (104) and the formulae of Bianchi is due to the definition of the angle σ.

† L. c., p. 439; German trans., 467.

repetition of the same transformation, that is, the constant c must have the same value also. Otherwise, the factor $\tan\left(\frac{\theta_1-\theta}{2}\right)$ doesn't vanish. For the latter case, we have $\phi=\omega+\pi$, which is an evident solution of the equations for S_1 similar to (88) for S.

It is our purpose now to study the theorem of permutability with regard to its bearing upon the transformations of the A-surfaces. We consider first the exceptional case which leads to the formula (105).

Let S be an A-surface with spherical representation (4) and denote by Σ the pseudospherical surface with the same spherical representation. Corresponding to the solution $\theta_1=\theta(u, v, \sigma_1, c_1)$ of the system (88), there exists a transform S_1', of the original surface, and a transform Σ_1 of Σ. The function ϕ given by (105) leads to a Bäcklund transformation of Σ_1 into a pseudospherical surface Σ_2. But we have seen that the function θ, by means of which a Bäcklund transformation is given, serves to determine a transformation of all the A-surfaces in the same group with the given pseudospherical surface into A-surfaces with the same spherical representation as this Bäcklund transform. Hence S_1' and all the A-surfaces with the same spherical representation are transformed by means of the function ϕ of (105) into surfaces of the same class as Σ_2.

We consider now the general case where ϕ is given by (103). As before, we denote by S_1' the transform of S by $\theta(u, v, \sigma_1, c_1)$, and by S_1'' the transform by means of $\theta(u, v, \sigma_2, c_2)$; for brevity, we write

$$\theta_1=\theta(u, v, \sigma_1, c_1), \quad \theta_2=\theta(u, v, \sigma_2, c_2).$$

Denote by Σ_1' and Σ_1'' the pseudospherical surfaces into which Σ is transformed by means of these respective functions. By the theorem of permutability, we know that if Σ_1' is transformed by means of ϕ and angle σ_2, we get the same surface Σ_2 as when Σ_1'' is transformed by means of ϕ and σ_1. But when S_1' is transformed by means of ϕ and σ_2 the transform S_2' corresponds with Σ_2 with parallelism of tangent planes; similarly, the surface S_2'', which is the transform of S_1'' by ϕ and σ_2. Hence, S_2' and S_2'' are A-surfaces of the same class. It remains to be discovered in what cases S_2' and S_2'' coincide. We proceed as in the case of pseudospherical surfaces.

Denote by λ' and μ' the values of λ and μ when S is transformed into S_1''; then, from (91),

$$\lambda' = (A \cos \omega + C \sin \omega) \sin \sigma_1, \quad \mu' = (-A \sin \omega + C \cos \omega) \sin \sigma_1. \quad (106)$$

In like manner, we denote by λ_1', μ_1' the corresponding functions for S_1', when the latter is transformed into S_2''; from (91) it follows that

$$\lambda_1' = (A_1' \cos \theta_1 + C_1' \sin \theta_1) \sin \sigma_2, \quad \mu_1' = (-A_1' \sin \theta_1 + C_1' \cos \theta_1) \sin \sigma_2, \quad (107)$$

where A_1' and C_1' are given by (93) when σ, θ, λ, μ are given the values $\sigma_1, \theta_1, \lambda', \mu'$.

In a manner similar to (80), we find for the projections, upon the tangents to the lines of curvature of S_1', of the line $M_1'M_2'$ the values

$$\lambda_1' \cos \phi - \mu_1' \cos \sigma_2 \sin \phi, \quad \lambda_1' \sin \phi + \mu_1' \cos \sigma_2 \cos \phi, \quad \mu_1' \sin \sigma_2.$$

As in the case of pseudospherical surfaces, we find that the tangents to the lines of curvature of S_1'' make with the directions MP, MQ, MN of S angles whose cosines are given by (98). Consequently, the coordinates of M_2' with respect to these directions for S as axes are

$$\begin{aligned}
\xi &= \lambda' + \lambda_1' \cos(\phi - \omega) - \mu_1' \cos \sigma_2 \sin(\phi - \omega), \\
\eta &= \mu' \cos \sigma_1 + \lambda_1' \cos \sigma_1 \sin(\phi - \omega) + \mu_1' \cos \sigma_1 \cos \sigma_2 \cos(\phi - \omega) - \sin \sigma_1 \sin \sigma_2 \mu_1', \\
\zeta &= \mu' \sin \sigma_1 + \lambda_1' \sin \sigma_1 \sin(\phi - \omega) + \mu_1' \sin \sigma_1 \cos \sigma_2 \cos(\phi - \omega) + \cos \sigma_1 \sin \sigma_2 \mu_1'.
\end{aligned}$$

And when we pass to the axes of the original trihedron associated with S and denote by x_2', y_2', z_2', the coordinates of M_2' with respect to these axes, we find

$$\left.\begin{aligned}
x_2' &= \lambda' \cos \theta_1 - \mu' \cos \sigma_1 \sin \theta_1 + (\lambda_1' \cos \theta_1 - \mu_1' \cos \sigma_1 \cos \sigma_2 \sin \theta_1) \cos(\phi - \omega) \\
&\quad - (\mu_1' \cos \sigma_2 \cos \theta_1 + \lambda_1' \cos \sigma_1 \sin \theta_1) \sin(\phi - \omega) + \sin \sigma_1 \sin \sigma_2 \mu_1' \sin \theta_1, \\
y_2' &= \lambda' \sin \theta_1 + \mu' \cos \sigma_1 \cos \theta_1 + (\lambda_1' \sin \theta_1 + \mu_1' \cos \sigma_1 \cos \sigma_2 \cos \theta_1) \cos(\phi - \omega) \\
&\quad - (\mu_1' \cos \sigma_2 \sin \theta_1 - \lambda_1' \cos \sigma_1 \cos \theta_1) \sin(\phi - \omega) - \sin \sigma_1 \sin \sigma_2 \mu_1' \cos \theta_1, \\
z_2' &= \mu' \sin \sigma_1 + \lambda_1' \sin \sigma_1 \sin(\phi - \omega) + \mu_1' \sin \sigma_1 \cos \sigma_2 \cos(\phi - \omega) + \cos \sigma_1 \sin \sigma_2 \mu_1'.
\end{aligned}\right\} (108)$$

The coordinates of S_2'', which we shall denote by x_2'', y_2'', z_2'', can be found in the same manner. They can be obtained from (108) by putting double prime for the

primes and interchanging the subscripts 1 and 2 of the θ's and σ's. For S_2' and S_2'' to be coincident, these corresponding coordinates must be equal, or what is the same thing, we must have

$$\begin{aligned} \cos\theta_1 (x_2' - x_2'') + \sin\theta_1 (y_2' - y_2'') &= 0, \\ \cos\theta_2 (x_2' - x_2'') + \sin\theta_2 (y_2' - y_2'') &= 0, \end{aligned} \quad z_2' = z_2''.$$

When the values for $x_2', \ldots, x_2'' \ldots, z_2''$ are put in this equation, they become

$$\left.\begin{aligned}
&[\lambda_1'' \cos(\theta_2 - \theta_1) - \mu_1'' \cos\sigma_1 \cos\sigma_2 \sin(\theta_2 - \theta_1) - \lambda_1'] \cos(\phi - \omega) \\
&\qquad + [\mu_1' \cos\sigma_2 - \mu_1'' \cos\sigma_1 \cos(\theta_2 - \theta_1) \\
&\qquad - \lambda_1'' \cos\sigma_2 \sin(\theta_2 - \theta_1)] \sin(\phi - \omega) \\
&\qquad = \lambda' - \lambda'' \cos(\theta_2 - \theta_1) + \mu'' \cos\sigma_2 \sin(\theta_2 - \theta_1) \\
&\qquad - \sin\sigma_1 \sin\sigma_2 \mu_1'' \sin(\theta_2 - \theta_1), \\
&[\lambda_1' \cos(\theta_2 - \theta_1) + \mu_1' \cos\sigma_1 \cos\sigma_2 \sin(\theta_2 - \theta_1) - \lambda_1''] \cos(\phi - \omega) \\
&\qquad + [\mu_1'' \cos\sigma_1 - \mu_1' \cos\sigma_2 \cos(\theta_2 - \theta_1) \\
&\qquad + \lambda_1' \cos\sigma_1 \sin(\theta_2 - \theta_1)] \sin(\phi - \omega) \\
&\qquad = \lambda'' - \lambda' \cos(\theta_2 - \theta_1) - \mu' \cos\sigma_1 \sin(\theta_2 - \theta_1) \\
&\qquad + \sin\sigma_1 \sin\sigma_2 \mu_1' \sin(\theta_2 - \theta_1), \\
&[\mu_1' \sin\sigma_1 \cos\sigma_2 - \mu_1'' \sin\sigma_2 \cos\sigma_1] \cos(\phi - \omega) \\
&\qquad + [\lambda_1' \sin\sigma_1 - \lambda_1'' \sin\sigma_2] \sin(\phi - \omega) \\
&\qquad = \sin\sigma_2 (\mu'' - \mu_1' \cos\sigma_1) - \sin\sigma_1 (\mu' - \mu_1'' \cos\sigma_2).
\end{aligned}\right\} \quad (109)$$

When the λ's, μ's, $\sin(\phi - \omega)$ and $\cos(\phi - \omega)$ are given the values determined by (106), (107) and (102), these equations take the forms

$$\begin{aligned}
&L_1\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + M_1\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) + N_1 (A\cos\omega + C\sin\omega) \\
&\qquad\qquad + R_1(-A\sin\omega + C\cos\omega) = 0, \\
&L_2\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + M_2\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) + N_2 (A\cos\omega + C\sin\omega) \\
&\qquad\qquad + R_2(-A\sin\omega + C\cos\omega) = 0, \\
&L_3\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + M_3\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) = 0,
\end{aligned}$$

where $L_1, \ldots, M_3$ are functions independent of A, C and ω. But when these various functions are calculated, it is found that they are all zero, so that in every

case S_2' and S_2'' coincide; we call it S_3. Hence the generalized theorem of permutability for A-surfaces.

When the A-surface, which is the transform by means of θ_1 of a given A-surface, S, is transformed by means of ϕ, given by (103), *and the angle σ_2, the new surface is the same as the transform by means of ϕ and angle σ_1 of the surface into which S is transformed by means of θ_2.*

We have remarked that, when ω is a solution of equation (7), $\omega + \pi$ also is a solution. As θ_1 and θ_2 in (103) satisfy the relation $\theta_1 + \theta_2 \neq (2m+1)\pi$, in order that ϕ may have the value $\omega + \pi$ it is necessary that

$$\sigma_2 = \sigma_1 + 2m\pi.$$

When these values are substituted in (96) and (100), both of the latter reduce to (88) in which θ is replaced by θ_1 and θ_2 respectively, and

$$\sigma = \sigma_1 = \sigma_2 - 2\pi m.$$

Thus all the equations of condition are satisfied for this value of ϕ, and, furthermore, it is in no way dependent upon θ_1 and θ_2. We shall show that this is true also of the surface S_3 which evidently belongs to the same group of surfaces as the original surface S.

For the sake of brevity, we put

$$a = \sin^2\sigma_1\left(\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u}\right), \quad b = \sin^2\sigma_1\left(\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v}\right).$$

The function λ_1 and μ_1 for the transformations of S are

$$\lambda_1 = (A\cos\omega + C\sin\omega)\sin\sigma_1, \quad \mu_1 = (-A\sin\omega + C\cos\omega)\sin\sigma_1,$$

and the functions λ_1', μ_1' for the transformation of S_1 are

$$\begin{aligned} \lambda_1' &= a\cos\theta_1 + b\sin\theta_1 + \lambda_1, \\ \mu_1' &= -a\sin\theta_1 + b\cos\theta_1 + \mu_1\cos\sigma_1. \end{aligned}$$

When these values are substituted in (108), we get for the projections of MM_3 upon the axes of the trihedron of S the values

$$-a, \quad -b, \quad (-A\sin\omega + C\cos\omega)\sin^2\sigma.$$

Hence if x, y, z denote the coordinates of a surface S with the linear element (1) and the spherical representation of its lines of curvature determined by ω, the surface whose coordinates are of the form

$$x - \sin^2\sigma\left(\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u}\right)X_1 - \sin^2\sigma\left(\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v}\right)X_2 + (-A\sin\omega + C\cos\omega)\sin^2\sigma X \qquad (109')$$

is an A-surface with the same representation of its lines of curvature. Hence, when one has given an A-surface, one can find at once an infinity of A-surfaces with the same representation of their lines of curvature as those of the given surface; their coordinates are of the above form in which σ is constant for each surface, but varies with the surface.

An exception to the foregoing arises in the case of pseudospherical surfaces, for when

$$A = \cos\omega, \quad C = \sin\omega,$$

the transform is the same as the original surface. Conversely, this is possible only in case

$$\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} = 0, \quad \frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} = 0, \quad -A\sin\omega + C\cos\omega = 0.$$

If we replace the last of these equations by

$$A = t\cos\omega, \quad C = t\sin\omega,$$

it is found from the first that t is constant; hence, the surface must be pseudospherical.

If the surface is a parallel of a pseudospherical surface, then (23),

$$A = \cos\omega - a\sin\omega, \quad C = \sin\omega + a\cos\omega,$$

where a is a constant. For this case the coordinates of the transforms are

$$x + a\sin^2\sigma X, \quad y + a\sin^2\sigma Y, \quad z + a\sin^2\sigma Z.$$

Hence the transforms are the surfaces parallel to the given surface, and when $\sigma = \pi/2$, the pseudospherical surface of the group is defined.

We consider, furthermore, the case where the given surface is a surface of Bianchi, defined by (56) or (64). In consequence of the relations (46) and (50), one finds for the coordinates of the transforms after the above manner

$$x\cos^2\sigma\,,\quad y\cos^2\sigma\,,\quad z\cos^2\sigma\,.$$

Hence, all the transforms are homothetic to the given surface, and when $\sigma=\pi/2$ the origin is the transform.

It was seen that the transforms of the parallels of pseudospherical surfaces are not homothetic to the original surface, hence it is of interest to seek the conditions to be satisfied in order that the transforms be homothetic to the given surface.

From (109′) it is seen that the necessary and sufficient condition for this is that the coordinates of the surface be of the form

$$\begin{aligned}
x &= \left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)X_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)X_2+(A\sin\omega-C\cos\omega)\,X,\\
y &= \left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)Y_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)Y_2+(A\sin\omega-C\cos\omega)\,Y,\\
z &= \left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)Z_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)Z_2+(A\sin\omega-C\cos\omega)\,Z,
\end{aligned}$$

when the linear element is

$$ds^2=A^2du^2+C^2dv^2.$$

The surface with the above coordinates is generated by the point whose coordinates are

$$\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u},\quad \frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v},\quad A\sin\omega-C\cos\omega$$

with respect to the trihedron with fixed vertex and which rotates so as to have its axes parallel to the axes of the trihedron of any A-surface whose spherical representation is determined by the same ω. The projections on these axes of a displacement are given by (40′), when $\lambda\cos\theta$, $\lambda\sin\theta$, μ are replaced by the above value. This gives

$$\left[\frac{\partial}{\partial u}\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)-\frac{\partial\omega}{\partial v}\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+\sin\omega\,(A\sin\omega-C\cos\omega)\right]du+0\,.\,dv,$$

$$0\,.\,du+\left[\frac{\partial}{\partial v}\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+\frac{\partial\omega}{\partial u}\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)-\cos\omega\,(A\sin\omega-C\cos\omega)\right]dv,$$

$$0\;du+0\;dv\,.$$

From the conditions of our problem the coefficient du in the first must be A and of dv in the second C. Put

$$\phi = A \cos\omega + C \sin\omega,$$

then the equations of condition become

$$\frac{\partial^2\phi}{\partial u^2} - \frac{\partial \log\cos\omega}{\partial u}\frac{\partial\phi}{\partial u} - \frac{\partial \log\sin\omega}{\partial v}\frac{\partial\phi}{\partial v} - \cos^2\omega.\phi = 0,$$

$$\frac{\partial^2\phi}{\partial v^2} - \frac{\partial \log\cos\omega}{\partial u}\frac{\partial\phi}{\partial u} - \frac{\partial \log\sin\omega}{\partial v}\frac{\partial\phi}{\partial v} - \sin^2\omega.\phi = 0.$$

In consequence of the relation

$$A\frac{\partial\omega}{\partial u} = \frac{\partial C}{\partial u},$$

when a common solution of the above equations is known, the function C is given by the linear equation

$$\frac{\partial C}{\partial u}\cos\omega + \sin\omega\frac{\partial\omega}{\partial u}, \quad C = \phi\frac{\partial\omega}{\partial u},$$

so that its determination requires two quadratures and then A is found directly.

§9.—*General Transformations of Surfaces of Bianchi.*

It was seen that the surfaces of Bianchi of the three types had for transforms of angle $\pi/2$ surfaces of the parabolic type. We consider now the transforms for any value of σ. And we begin by remarking that, when Σ is of the parabolic type and the origin is the point through which all of the associated circles pass, the coefficients of the linear element may be given the form (58),

$$A = -(e^{-\alpha_1}\cos\omega - \gamma_1\sin\omega), \quad C = -(e^{-\alpha_1}\sin\omega + \gamma_1\cos\omega), \tag{58}$$

where the expressions for α_1 and γ_1 are given by (46) and (50). If we effect upon Σ a transformation of angle σ and by means of θ, given by (88), the values of λ and μ are

$$\lambda = -e^{-\alpha_1}\sin\sigma, \quad \mu = -\gamma_1\sin\sigma,$$

and the coefficients of the linear element of the transform Σ_1 are found from (93)

to have the form

$$\left.\begin{aligned} A_1 &= \sin\sigma\, e^{-a_1} \cos\theta_1 - e^{-a_1}\cos\theta + \gamma_1 \cos\sigma \sin\theta, \\ C_1 &= \sin\sigma\, e^{-a_1}\sin\theta_1 - e^{-a_1}\sin\theta - \gamma_1\cos\sigma\cos\theta. \end{aligned}\right\} \tag{110}$$

From the general defining theorem of Bianchi, it follows that if one of the transforms of Σ_1, for σ a right angle, reduces to a point, then Σ_1 is a surface of Bianchi of the parabolic type.

For such a transformation the value of λ_1 is

$$\lambda_1 = \sin\sigma\, e^{-a_1}\cos(\theta_1 - \theta) - e^{-a_1},$$

and the coefficients of the linear element of transform are found by (44) to be

$$\begin{aligned} A_2 &= e^{-a_1}\{\cos\omega\cos(\theta_1-\theta) - \cos\sigma\sin\omega\sin(\theta_1-\theta) - \sin\sigma\cos\omega + \lambda_1 e^{a_1}\cos\phi\}, \\ C_2 &= e^{-a_1}\{\sin\omega\cos(\theta_1-\theta) + \cos\sigma\cos\omega\sin(\theta_1-\theta) - \sin\sigma\sin\omega + \lambda_1 e^{a_1}\sin\phi\}, \end{aligned}$$

where ϕ is a solution of system (48). For one of these transforms to reduce to a point there must be a solution ϕ of (48) which makes both of the above expressions zero. Multiply the first by $\cos\omega$, the second by $\sin\omega$ and add; similarly multiply by $-\sin\omega$, $\cos\omega$ respectively, and add; this gives

$$\left.\begin{aligned} [\sin\sigma\cos(\theta_1-\theta) - 1]\cos(\phi-\omega) &= \sin\sigma - \cos(\theta_1-\theta), \\ [\sin\sigma\cos(\theta_1-\theta) - 1]\sin(\phi-\omega) &= -\cos\sigma\sin(\theta_1-\theta). \end{aligned}\right\} \tag{111}$$

When these are compared with (102), it is found that the latter take the above form when σ_2 is $\pi/2$ and θ_1 is the same as θ_2 in (102). Hence:

When a surface of Bianchi of the parabolic type is transformed under any angle σ, the transform is a surface of the same kind. And if θ and σ are the functions defining the transformation, and θ_1 is the function appearing in the expression for the coefficients of the linear element (58), the function ϕ which leads to the point surface of the transforms of the new surface is given by (103) in the form

$$\tan\left(\frac{\phi-\omega}{2}\right) = \frac{\sin\left(\frac{\pi}{4}+\frac{\sigma}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\sigma}{2}\right)}\tan\left(\frac{\theta_1-\theta}{2}\right).$$

Let Σ be a surface of Bianchi of any type whatever. Denote by S_1 its transform of angle σ and function θ_1 and by Σ_1 its transform by θ' where the latter is

a solution of equations (38). From preceding results we know that Σ_1 is a surface of Bianchi of the parabolic type. The foregoing theorem tells us that, if we transform Σ_1 under angle σ and by means of ϕ given by (102) which now take the form

$$\left.\begin{aligned} \cos(\phi-\omega) &= \frac{\sin\sigma-\cos(\theta'-\theta)}{\sin\sigma\cos(\theta'-\theta)-1}, \\ \sin(\phi-\omega) &= -\frac{\cos\sigma\sin(\theta'-\theta)}{\sin\sigma\cos(\theta'-\theta)-1}, \end{aligned}\right\} \tag{112}$$

the resulting surface Σ_2 will be a surface of Bianchi of the parabolic type. But this surface Σ_2 is also the transform of S_1 for $\sigma=\pi/2$ and same ϕ. By varying θ and keeping θ fixed we get an infinty of surfaces Σ_2 of the parabolic type which are orthogonal to the circles associated with S_1. From the theorem of Bianchi it follows that one of the surfaces Σ_2 must reduce to a point in order that S_1 be a surface of Bianchi of the parabolic type. We procced to the determination of the cases where this is possible.

We begin with the general definition (64) of surfaces of Bianchi, whose linear element has for coefficients the squares of the quantities (66),

$$\left.\begin{aligned} A &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\cos\omega+t\sin\omega, \\ C &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\omega-t\cos\omega. \end{aligned}\right\} \tag{66}$$

The functions λ and μ, which figure in the transformation of angle σ of these surfaces, have the forms

$$\lambda=\tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\sigma, \quad \mu=-t\sin\sigma,$$

and the quantities A_1 and C_1 for the transform S_1 are found by (93) to be

$$\left.\begin{aligned} A_1 = \sin\sigma\,[\tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\cos\theta_1+\beta\sin\theta_1] \\ +\tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\cos\theta+t\cos\sigma\sin\theta, \\ C_1 = \sin\sigma\,[\tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\sin\theta_1-\beta\cos\theta_1] \\ +\tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\theta-t\cos\sigma\cos\theta. \end{aligned}\right\} \tag{113}$$

In order to get the surfaces orthogonal to the circles associated with S_1, we have only to transform S_1 under angle $\sigma=\pi/2$ and by ϕ, given by (112). For this

transformation λ_1 has the value

$$\lambda_1 = \sin\sigma\,[\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\}\cos(\theta_1 - \theta) + \beta\sin(\theta_1 - \theta)] \\ + \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)\,e^{-\alpha}\}. \qquad (114)$$

In calculating the functions A_2 and C_2 for the new surface, by means of formulæ similar to (44), it must be remembered that θ_1 satisfies equations (38) and θ equations (88). These functions are found to be of the form

$$\begin{aligned} A_2 = {} & [\sin\sigma\,\tfrac{1}{2}\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\} + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\}\cos(\theta_1 - \theta) \\ & + \beta\sin(\theta_1 - \theta)]\cos\omega - \cos\sigma\,[\tfrac{1}{2}\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\}\sin(\theta_1 - \theta) \\ & - \beta\cos(\theta_1 - \theta)]\sin\omega + \lambda_1\cos\phi, \\ C_2 = {} & [\sin\sigma\,\tfrac{1}{2}\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\} + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\}\cos(\theta_1 - \theta) \\ & + \beta\sin(\theta_1 - \theta)]\sin\omega + \cos\sigma\,[\tfrac{1}{2}\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\}\sin(\theta_1 - \theta) \\ & - \beta\cos(\theta_1 - \theta)]\cos\omega + \lambda_1\sin\phi. \end{aligned}$$

For one of these surfaces to reduce to a point, it must be possible to find a function ϕ which makes A_2 and C_2 zero. Equate the above expressions to zero; multiply by $\cos\omega$ and $\sin\omega$ respectively, and add; again multiply by $-\sin\omega$, $\cos\omega$ and add; this gives

$$\left.\begin{aligned} & \lambda_1\cos(\phi - \omega) + \sin\sigma\,\tfrac{1}{2}\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\} \\ & \qquad + \tfrac{1}{2}\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\}\cos(\theta_1 - \theta) + \beta\sin(\theta_1 - \theta) = 0, \\ & \lambda_1\sin(\phi - \omega) + \cos\sigma\,[\tfrac{1}{2}\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\}\sin(\theta_1 - \theta) \\ & \qquad - \beta\cos(\theta_1 - \theta)] = 0. \end{aligned}\right\} \quad (115)$$

Expressing the condition that

$$\sin^2(\phi - \omega) + \cos^2(\phi - \omega) = 1,$$

it is found that we must have k equal to zero. Hence the theorem:

The surfaces of Bianchi of any type are transformed into surfaces of the parabolic type when σ is $\pi/2$, and only surfaces of the parabolic type have such transforms for any value of σ.

It only remains to point out the transformation function θ' which gives ϕ such a form that Σ_2 reduces to a point. If we substitute in (115) for $\cos(\phi - \omega)$

and sin $(\phi - \omega)$ the values (112), the former can be reduced to

$$\left.\begin{aligned}&\tfrac{1}{2}\{e^{\sigma} + (\beta^2 + k)\,e^{-\sigma}\}\cos(\theta' - \theta) + \tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\cos(\theta_1 - \theta)\\&\qquad\qquad\qquad\qquad + \beta\sin(\theta_1 - \theta) = 0,\\&\sin\sigma\,[\tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\sin(\theta_1 - \theta') - \beta\cos(\theta_1 - \theta')]\\&\qquad - \tfrac{1}{2}\{e^{\sigma} - e^{-\sigma}(\beta^2 + k)\}\sin(\theta_1 - \theta)\\&\qquad - \tfrac{1}{2}\{e^{\sigma} + e^{-\sigma}(\beta^2 + k)\}\sin(\theta' - \theta) + \beta\cos(\theta_1 - \theta) = 0.\end{aligned}\right\} \quad (116)$$

Since this must be true for all values of σ, the last equation reduces to the two

$$\left.\begin{aligned}&\tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\sin(\theta_1 - \theta') - \beta\cos(\theta_1 - \theta') = 0,\\&\tfrac{1}{2}\{e^{\sigma} + (\beta^2 + k)\,e^{-\sigma}\}\sin(\theta' - \theta) + \tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\sin(\theta_1 - \theta)\\&\qquad\qquad\qquad\qquad - \beta\cos(\theta_1 - \theta) = 0.\end{aligned}\right\} \quad (117)$$

If the first of (116) be multiplied by $\cos\theta$ and the last of (117) by $\sin\theta$ and the resulting equations subtracted, and the former of these be multiplied by $\sin\theta$ and the latter by $\cos\theta$ and the results added we get

$$\left.\begin{aligned}&\tfrac{1}{2}\{e^{\sigma} + (\beta^2 + k)\,e^{-\sigma}\}\cos\theta' + \tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\cos\theta_1 + \beta\sin\theta_1 = 0,\\&\tfrac{1}{2}\{e^{\sigma} + (\beta^2 + k)\,e^{-\sigma}\}\sin\theta' + \tfrac{1}{2}\{e^{\sigma} - (\beta^2 + k)\,e^{-\sigma}\}\sin\theta_1 - \beta\cos\theta_1 = 0.\end{aligned}\right\} \quad (118)$$

When these values are substituted in the first of (117), it is satisfied.

A comparison of these results with the discussion of the transformations of surfaces of Bianchi, developed in §6, will reveal the fact that the function θ', which gives ϕ such a form that Σ_2 shall be a point, is the very function which transforms Σ of the parabolic type into a point. A similar result obtains in the discussion of (111).

§10.—*Transformations of a Particular Class of Surfaces.*

The surfaces to be considered in this section are suggested by a similar class of pseudospherical surfaces studied by Darboux* and Bianchi.† We start with the remark that an evident solution of equation (7) is $\omega = 0$. From the general expression for the linear element of pseudospherical surfaces

$$ds^2 = \cos^2\omega du^2 + \sin^2\omega dv^2,$$

* Leçons, Vol. 3, p. 468.

† Lezioni, p. 440; Ger. trans., 466.

it follows that the corresponding surface of this kind for the above solution is a curve of parameter u. We take the very special case where this curve is a straight line, and let it be the z-axis, then the rectangular coordinates of the surface are

$$x = 0, \quad y = 0, \quad z = u.$$

The tangent planes to this degenerate surface pass through the z-axis, hence the tangent planes to the other A-surfaces belonging to the solution $\omega = 0$ are cylinders whose generators are parallel to this line. Consequently, the most general surfaces of this group are given by

$$x = V\cos v, \quad y = V\sin v, \quad z = u,$$

where V is a function of v alone whose form will be determined later. The linear element of the surface has the form

$$ds^2 = du^2 + (V'^2 + V^2)\,dv^2,$$

where the accent denotes differentiation. From the above expressions we get for the direction-cosines of the normal to the surface

$$X = -\frac{V'\sin v + V\cos v}{\sqrt{V^2 + V'^2}}, \quad Y = \frac{V'\cos v - V\sin v}{\sqrt{V^2 + V'^2}}, \quad Z = 0, \tag{119}$$

and denoting by X_1, Y_1, Z_1; X_2, Y_2, Z_2 the direction-cosines of the tangents to the lines of curvature $v = \text{const.}$, $u = \text{const.}$ respectively, we have

$$\left.\begin{aligned} &X_1 = 0, \quad Y_1 = 0, \quad Z_1 = 1, \\ &X_2 = \frac{V'\cos v - V\sin v}{\sqrt{V^2 + V'^2}}, \quad Y_2 = \frac{V'\sin v + V\cos v}{\sqrt{V^2 + V'^2}}, \quad Z_2 = 0. \end{aligned}\right\} \tag{120}$$

From (119) we have

$$\left.\begin{aligned} \frac{\partial X}{\partial v} &= \frac{(V^2 + 2V'^2 - VV'')(V\sin v - V'\cos v)}{(V^2 + V'^2)^{\frac{3}{2}}}, \\ \frac{\partial Y}{\partial v} &= -\frac{(V^2 + 2V'^2 - VV'')(V\cos v + V'\sin v)}{(V^2 + V'^2)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{121}$$

But from (4), it is seen that for this case we must have

$$\Sigma\left(\frac{\partial X}{\partial v}\right)^2 = 1,$$

so that V must satisfy the condition

$$[V^2 + 2V'^2 - VV'']^2 = (V^2 + V'^2)^2.$$

A comparison of (120), (121) and (55) shows that it may be replaced by

$$V^2 + 2V'^2 - VV'' = V^2 + V'^2,$$

from which we have

$$V = ae^{cv}.$$

Hence the A-surfaces of the same class as the z-axis are given by

$$x = ae^{cv}\cos v, \quad y = ae^{cv}\sin v, \quad z = u, \tag{122}$$

where a and c are arbitrary constants, and the linear element is

$$ds^2 = du^2 + a^2e^{2cv}(1 + c^2)\,dv^2. \tag{123}$$

For the present case, the equations (88) reduce to

$$\frac{\partial\theta}{\partial u} = \frac{\sin\theta}{\sin\sigma}, \quad \frac{\partial\theta}{\partial v} = \frac{\cos\sigma}{\sin\sigma}\sin\theta,$$

of which the general integral is

$$\tan\frac{\theta}{2} = Ce^{\frac{u + \cos\sigma\,.\,v}{\sin\sigma}},$$

where C is an arbitrary constant. When C is zero, θ is zero or π, and hence gives the surface from which we start, we take C equal to unity. From (123) we have

$$\lambda \sin\sigma, \quad \mu = a\sqrt{1 + c^2}\,e^{cv}\sin\sigma,$$

so that, by means of (80), (119) and (120), we find for the coordinates of the transforms of these surfaces

$$\left.\begin{aligned}
x_1 &= ae^{cv}\cos v + \frac{c\cos v - \sin v}{\sqrt{1 + c^2}}\,\frac{\sin v}{\cosh\alpha} \\
&\qquad - ae^{cv}\sin\sigma\,[(c\cos v - \sin v)\cos\sigma\tanh\alpha \\
&\qquad + \sin\sigma\,(c\sin v + \cos v)], \\
y_1 &= ae^{cv}\sin v + ae^{cv}\sin\sigma\,[(c\cos v - \sin v)\sin\sigma \\
&\qquad - \cos\sigma\tanh\alpha\,(c\sin v + \cos v)] + \frac{c\sin v + \cos v}{\sqrt{1 + c^2}}\,\frac{\sin\sigma}{\cosh\alpha}, \\
z_1 &= u - \sin\sigma\tanh\alpha - \frac{a\sqrt{1 + c^2}\,e^{cv}\sin\sigma\cos\sigma}{\cosh\alpha},
\end{aligned}\right\} \tag{124}$$

where we have put, for the sake of brevity,

$$\alpha = \frac{u + \cos\sigma v}{\sin\sigma}.$$

When in (121) we put c equal to zero, these formulæ define circular cylinders, so that (124) define all the transforms of circular cylinders. Furthermore, when we put a also equal to zero, we have the original case, the z-axis; from (124) we have for the formulæ of its transforms

$$x_1 = -\frac{\sin v \sin\sigma}{\cosh\alpha}, \quad y_1 = \frac{\cos v \,.\, \sin\sigma}{\cosh\alpha}, \quad z_1 = u - \sin\sigma \tanh\alpha. \qquad (125)$$

Bianchi has shown* that the surfaces so defined are pseudospherical helicoidal surfaces whose meridian profile in the tractrix.

If, in particular, we put σ equal to $\pi/2$ (125), define the pseudosphere.† In this case α and, therefore, θ is a function of u alone, so that the surfaces defined by

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{1+c^2}\cosh u}(c\cos v - \sin v) - ace^{cv}\sin v, \\ y_1 &= \frac{1}{\sqrt{1+c^2}\cosh u}(c\sin v + \cos v) + ace^{cv}\cos v, \\ z_1 &= u - \tanh u, \end{aligned}$$

are *moulure* surfaces.

The complete determination of all the A-surfaces with the same spherical representation of their lines of curvature, as the surfaces defined by (124) requires the integration of the equation of the form (8), which, in this case, reduces to

$$\frac{\partial^2\psi}{\partial u\,\partial v} + \frac{e^{2\alpha}-1}{1+e^{2\alpha}}\,\frac{\cos\sigma}{\sin\sigma}\,\frac{\partial\psi}{\partial u} + \frac{4e^{2\alpha}}{1-e^{4\alpha}}\,\frac{\partial\psi}{\partial v} = 0.$$

It is readily found that the invariant h‡ for this equation is zero, so that the complete integral can be found at once; it is

$$\psi\,(e^{\alpha} - e^{-\alpha}) = \int (e^{\alpha} - e^{-\alpha})\,V dv + U,$$

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* Lezioni, p. 441; Ger. trans., p. 467. † Bianchi, l. c., p. 443; Darboux, Leçons, Vol. 3, p. 465.
‡ Darboux. Leçons, Vol. 2, 28.

when U and V are arbitrary functions of u and v respectively. When this quadrature has been effected for each form of V, the further determination of these surfaces is direct and without integration. In consequence of the theorem of permutability, the transforms of all these surfaces can be found by algebraic processes and differentiation. Thus, if we effect upon the original surfaces (122) the transformation given by

$$\tan\frac{\theta_1}{2} = e^{\alpha_1}, \quad \alpha_1 = \frac{u + \cos\sigma_1 v}{\sin\sigma_1},$$

the functions Φ, which enter into the transformations of the resulting surfaces are given by (103), which now take the form

$$\tan\frac{\Phi}{2} = \frac{\sin\left(\frac{\sigma_1+\sigma}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma}{2}\right)}\frac{e^{\alpha_1}-e^{\alpha}}{1+e^{\alpha_1+\alpha}},$$

and for the particular case, $\sigma = \sigma_1$, we have by (105),

$$\tan\frac{\Phi}{2} = \frac{c' - u\cos\sigma_1 - v}{\sin\sigma_1\cos\alpha_1}.$$

The further transformations of the surfaces obtained by the application of the foregoing formulæ are merely a repetition of the preceding, so that all of the surfaces arising from (122) by transformations can be found directly.

It has been seen that certain moulure surfaces belong to the suite of surfaces which are given by the preceding transformations. In seeking for the transforms of these, we propose the more general question as to whether moulure surfaces can be transformed into surfaces of the same kind. To this end we take ω, a function of u alone. From the form, to which the first of equations (88) reduces, it is evident that for S_1 to be a moulure surface θ must be a function of u alone. Then these equations reduce to

$$\left.\begin{aligned}\sin\sigma\frac{\partial\theta}{\partial u} &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega,\\ \sin\sigma\frac{\partial\omega}{\partial u} &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega.\end{aligned}\right\} \tag{126}$$

From (17) it follows that ω and θ must satisfy also the equations

$$\frac{d\theta}{du} = \sqrt{\sin^2\theta - a^2}, \quad \frac{d\omega}{du} = \sqrt{\sin^2\omega - a_1^2}, \tag{127}$$

where a and a_1 are constants. If equations (126) be squared and the values for $\frac{d\theta}{du}$ and $\frac{d\omega}{du}$ from (127) be substituted, we get two equations from which, by subtraction, we find that a_1^2 is equal to a^2. The first of these equations is

$$\sin^2\sigma(\sin^2\theta - a^2) = (\sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega)^2. \tag{128}$$

If this equation be differentiated with respect to u and the values of $\frac{d\theta}{du}$ and $\frac{d\omega}{du}$ given by (126) be substituted, the equation is an identity. When $\sigma = \pi/2$, the above equation reduces to

$$\sin^2\theta\sin^2\omega = a^2.$$

Hence:

A moulure surface can always be transformed into a moulure surface and the transformation can be effected by algebraic processes.

It is clear to be sure that other A-surfaces can be transformed into moulure surfaces, just as all the transforms of a moulure surface are not surfaces of this kind.

PRINCETON UNIVERSITY.

On the Forms of Sextic Scrolls Having no Rectilinear Directrix.

BY VIRGIL SNYDER.

In my papers on sextic scrolls which appeared in Volume XXV of the Journal, two families of scrolls were omitted, namely, those whose generators are bisecants of a twisted cubic or quartic curve and those having a tacnodal directrix curve, analogous to those having a tacnodal directrix line, types 58-66, in my paper in the Journal, Volume XXVII, p. 77.

The present paper will discuss these two types and supply various other minor omissions of the previous papers. The list of types here found, together with those published in the preceding papers, is believed to be complete. The method will be the same as that employed before, namely, the establishment of a multiple correspondence between the points of two curves and drawing a generator between them.

§1.—S_6 *Having Two Distinct Double Conics.*

1. In type XIX (p. 90, Vol. XXV), the cubic is a plane nodal cubic, and all three conics pass through two common points. If in the correspondence there defined, $\mu = 0$ corresponds to 0 and ∞, and if $\lambda = 0$ corresponds to 0 and ∞, the line of intersection of the planes of the two conics becomes a double generator. The residual nodal curve is now a twisted quartic. The symbol is $2c_2^2+c_4^2+g^2$ (1). If the (2,2) correspondence has a double element, the surface becomes unicursal and has two double generators: $2c_2^2 + c_4^2 + 2g^2$ (2). These two forms are self-dual. The correspondence given in the Bulletin of the American Math. Society, Volume IX, p. 240, defines the S_6 of genus 1, having the symbol $c_{3,2}^2+3c_2^2$. The

three conics have two points in common. If, in this correspondence, a double element be introduced, a double generator appears; the symbol becomes $3c_2^2+c_{3,2}^2+g^2$. (Type LVIII, Vol. XXV, p. 81). If the plane determined by g^2 and the tangent to c_1 at their point of intersection also contains the tangent to c_2 at its point of intersection with g^2, then the latter counts for two double generators and the residual nodal curve is a fourth double conic, cutting $\overline{2g^3}$ twice and each of the preceding conics once (3). $4c_2^2+\overline{2g^3}$.

2. Any two conics which touch each other but lie in different planes may be written in the form:

$$c_1:\quad x=1,\quad y=\mu,\quad z=0,\quad w=\mu^2,$$
$$c_2:\quad x=0,\quad y=\lambda,\quad z=1,\quad w=\lambda^2.$$

The equations of a line joining λ to μ are

$$\mu x+\lambda z-y=0,\quad \mu^2x+\lambda^2z-w=0.$$

When the form of the correspondence between λ,μ is

$$c\lambda^2+d\mu^2+c\lambda\mu+f\lambda+g\mu+h=0$$

the surface is an S_6 having the common tangent $x=0$, $g=0$ for tacnodal generator. The equation is

$$\begin{vmatrix} cx-dz-ez & ey+fx-gz & dw+gy+hx & 0 \\ 0 & cx-dz-cz & ey+fx-gz & dw+gy+hx \\ z^2+zx & -2yz & y^2-xw & 0 \\ 0 & z^2+zx & -2yz & y^2-xw \end{vmatrix}=0.$$

The plane $cx-dz=0$ contains x^4 and a simple conic. The residual nodal curve is a nodal quartic, the node being at the point of tangency of the double conics. $2c_2^2+c_{4,2}^2+\overline{2g^2}$ (4). This surface belongs to a linear complex when $e=0$.

3. Given two conics $c_1(\lambda)$ and $c_2(\mu)$ which intersect in one point, (0, 0). Let λ,μ be in (2, 2) correspondence with (0, 0) as self-corresponding double point. Let

$$c_1:\quad x=\mu,\quad y=\mu^2,\quad z=0,\quad w=1,$$
$$c_2:\quad x=0,\quad y=\lambda^2,\quad z=\lambda,\quad w=1-\kappa\lambda^2.$$

The equations of a generator are

$$y - \mu x = \lambda z, \quad \lambda x - \lambda\mu w = (\kappa\lambda^2 - 1)\,\mu z.$$

The general residual nodal curve is a sextic having four double points at the points of intersection with the two conics. $2c_2^2 + c_{6,2}^2\,(5)$. The plane $x = 0$ of c_2 cuts c_1 in $\mu = 0$ and in $\mu = \infty$. If λ_1, λ_2 correspond to $\mu = \infty$, and $\lambda_1, \lambda_2, \infty$ are collinear, then $1 + \kappa\lambda_1\lambda_2 = 0$.

If the $(2, 2)$ correspondence be

$$a\lambda^2\mu^2 + b\lambda^2\mu + c\lambda\mu^2 + d\lambda^2 + e\lambda\mu + f\mu^2 = 0,$$

then $a\lambda^2 + c\lambda + f = 0$ has λ_1, λ_2 for roots. Hence, if $f\kappa + a = 0$, S_6 has a double generator $x = 0, fw - cz = 0$. The residual curve is c_5^2 with two double points on c_1 and four simple points on c_2. $2c^2 + c_{5,2}^2 + g^2\,(6)$. Similarly, the plane $z = 0$ cuts c_2 in $\lambda = 0$ and $\lambda = \infty$. When $\lambda = \infty$, the two equations of a generator become identical, but, by eliminating z, we have

$$\mu\,(y - \mu x)(\kappa\lambda^2 - 1) = \lambda^2\,(x - \mu w).$$

Hence $\mu_1\mu_2\kappa = 1$. Since μ_1, μ_2 are roots of the equation $a\mu^2 + b\mu + d = 0$, it follows that $f + d = 0$. In this case the surface has a second double generator $z = 0,\ ay + dw + bx = 0$. The residual is a c_4^2 of the second kind. $2c_2^2 + c_4^2 + 2g^2\,(7)$. By writing $-ay - bx + cz - dw \equiv u$, the equation of the scroll may be written

$$\begin{vmatrix} az^2 & z\,(ay - dz) & d\,(wy - x^2 + z^2) & -dz^2y \\ 0 & 0 & y\,(u + ay) & dwy - z\,(cy + ex) \\ u & cy + ex & -dy & 0 \\ 0 & u & -(cy + ex) & dyz \end{vmatrix} = 0.$$

The residual curve in the plane $u - mz = 0$ is a trinodal quartic. By properly choosing m, values of a, b, c, d, e can be found for which this quartic is a double conic. The residual nodal curve is then another c_2 in the plane. $x + m'(dw + cz) = 0$. $4c_2^2 + 2g^2\,(8)$. This type is self-dual. The new conics have one point in common with each other and establish a similar correspondence to that between λ and μ.

§2.—S_6 *Having Two Double Cubics.*

4. Consider the two cubics

$$x = 1 - \lambda^2, \quad y = \lambda(1 - \lambda^2), \quad z = 0, \quad w = 1,$$
$$x = 1 - \mu^2, \quad y = 0, \quad z = \mu(1 - \mu^2), \quad w = 1,$$

in which
$$(a + b)\lambda^2\mu^2 - a\lambda^2 - b\mu^2 = 0.$$

The point (1, 0, 0, 1) or (0, 0) is a double element, while (0, 0, 0, 1) counts for four simple self-corresponding elements, (1, 1), (1, — 1), (— 1, 1), (— 1, — 1). The residual curve is in general a quartic $\overline{2c^2_{7,2}} + c^2_4$ (9), but may consist of two conics by imposing the condition that the line joining λ to μ shall meet a conic. $2c^2_{5,2} + 2c^2_2$ (10). Exactly the same forms may exist for two twisted cubics which intersect in five points. One point should be taken for a double point and each of the others for simple self-corresponding points (11), (12).

§3.— *Various Forms.*

5. When the c_3 $\quad x = 1 - \lambda^2, \quad y = \lambda(1 - \lambda^2), z = 0, \quad w = 1,$
and c_2 $\quad x = \mu, \quad y = 0, \quad z = \mu^2 - \mu, w = 1,$

are connected by the relation

$$\lambda^2(a\mu^2 + b\mu + c) + (\mu - 1)[a'\lambda\mu - c(\mu - 1)] = 0,$$

the resulting S_6 has the symbol $c^2_{5,3} + c^2_2 + c^2_{3,2}$ (13). In the plane of the conic are two generators which intersect in the node of the cubic. If $a' = 0$ these two generators coincide. The symbol now becomes $c^2_4 + c^2_2 + c^2_{3,2} + g^2$ (14).

6. Given a unicursal c_4 and a twisted c_3 cutting c_4 in one node N and in two simple points P_1, P_2. Let the points λ of c_3 and μ of c_4 be in (2, 1) correspondence such that P_1, P_2 are simple self-corresponding points and N a 2, 1 point. The result is an S_6 of symbol $c^2_3 + c^2_7$ (15), c_7 has four double points.

The line P_1P_2 may be $\lambda_1\mu_1$ and also $\lambda_2\mu_2$, making the symbol $c^2_3 + c^2_6 + g^2$ (16). Each of the points in which g^2 cuts c_4 is on c_6.

7. If c_3 be nodal, c_7 will have a fourfold point at N. (17). The double generator may appear as before. (18). c_6 has a triple point and one other double point.

8. Consider the twisted quintic

$$x = \lambda^3, \quad y = \lambda^4, \quad z = \lambda^5, \quad w = 1,$$

having one double point and two cusps at the origin, and three apparent double points. The curve lies on the quadric cone $y^2 = xz$. The equations of a line joining the point λ to the point μ may be written

$$\begin{aligned} &z - \lambda y = \mu (y - \lambda x), \\ &x (\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3) - y (\lambda^2 + \lambda\mu + \mu^2) = \lambda^3\mu^3 w. \end{aligned}$$

If $\lambda = \alpha\mu$, this line will describe an S_6 having c_5 as a double curve. The residual is another c_5 having a triple point at (0, 0, 0, 1). When $\alpha = 1$, the surface reduces to the developable enveloped by the plane

$$\lambda^5 w + 10\lambda^2 x - 15\lambda y + 6z = 0.$$

This surface is not mentioned in published lists of sextic developables. $2c_{5,3}$ (19). Similarly, by making the same correspondence between the points of a c_5 with two nodes, which are the double points of the projective relation between λ and μ, another S_6 is defined, having two double quintics, each of which has two double points through which the other curve passes. $2c^2_{5,2}$ (20). A particular case is the known developable having a quintic for cuspidal edge. (Schwarz.)

§4.—S_6 *Having a Triple Cubic.*

9. Given the twisted cubic $x = \lambda, y = \lambda^2, z = \lambda^3, w = 1$. The equations of a line g connecting the point λ to the point μ are

$$x (\lambda + \mu) - \lambda\mu w = y, \quad y (\lambda + \mu) - \lambda\mu x = z,$$

from which
$$\lambda + \mu = \frac{xy - zw}{x^2 - yw}, \quad \lambda\mu = \frac{y^2 - zx}{x^2 - yw}.$$

If λ, μ be pairs of points in a quadratic involution, then g will generate a hyperboloid containing c_3 as a simple curve. Every generator of the same system as g will cut c_3 in two points which belong to the involution. Two generators will touch c_3; they correspond to the double points of the involution.

If λ, μ form a [2] involution, g will describe an S_4 having c_3 as double curve. If the [2] involution be cubic, S_4 has a simple rectilinear directrix. Planes through c_1 cut c_3 in a triad defined by the cubic involution. The double points of the involution are the pinch points of the surface. If the relation between λ and μ be $\lambda = \mu$, the quartic developable results.

An S_6 is defined when λ, μ define a [3] involution, on which c_3 is a triple curve. c_3^3 (21). If a double point exist on the cubic curve defined by $f_3\,(\lambda + \mu, \lambda\mu) = 0$, S_6 has a double generator and becomes unicursal. $c_3^3 + g^2$ (22). If the [3] involution be quartic, the plane containing two generators will contain a third. The points of c_3 are now arranged in quadruples belonging to the involution. A plane quartic curve cannot lie on the surface. A double generator may exist as before. When the involution is of the form

$$\lambda^2\mu^2 = \varkappa\,(\lambda + \mu)^3$$

the generator becomes cuspidal. An S_6 having three families of plane quartics is also defined by any (1, 2) relation between λ and μ. It always has a double generator.

10. In the same way, $S_{2(n-1)}$ can be generated by means of an involution $[n]$ between λ and μ, having c_3 for an $(n-1)$-fold curve. The genus is $\frac{(n-2)(n-3)}{2}$, and may be made less by the introduction of double points. Apart from double generators, c_3 is in every case the complete nodal curve. When the $[n]$ involution is an $(n+1)$-ic, a plane containing two generators will also contain a third; the envelope of these planes is a developable of order $\frac{n(n-1)}{2}$. Scrolls can be generated by similar involutions on every unicursal curve.*

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* E. Weyr, "Ueber Flächen sechsten Grades mit einer dreifachen cubischen Curve," Wiener Berichte, Vol. 85 (1882), has treated the case of a quartic involution synthetically. The condition for a cubic involution is given in Salmon's Geometry, 4th edition, p. 518. The S_6 defined by the quartic involution is self-dual, but not those defined by the general [3] involution.

§5.—S_6 *Having a Double Quartic.*

11. Various forms of the surface have already been found having a nodal quartic curve; the systematic discussion of all such surfaces will now be given. Let a rational c_4, depending on a parameter λ, be given. *If λ, μ two points on the curve be connected by a* $(1, 1)$ *correspondence, the lines joining λ to μ will define a rational S_6, having c_4 for a double curve.* The curve will be double, for if $\lambda (T) \mu$, then at any point K there is a point L such that $L (T) K$ and a point M such that $K (T) M$. To find the order of the surface, two cases will be considered, according as c_4 has or has not a node.

Let P be a node on c_4. There are two values of the parameter at this point, and to each correspond two generators, hence through P pass in general four generators. Project all the points of c_4 from P. The joining line will form a quadric cone K_2. The lines joining corresponding points will project into planes passing through P. Now cut K_2 by any plane π. The section of K_2 on π will be a conic, and the generators will project into lines connecting corresponding points of two projective ranges on c_2. These lines envelope another conic, hence all the generators of our scroll touch a quadric cone. The order of a scroll is determined by the number of generators which cut an arbitrary line. A line l through P will cut four generators at P. Two lines tangent to c_2' in π pass through the trace of l in π, hence l cuts six generators. Since the S_6 is such that every generator touches a quadric cone, it must be of the $(2, 4)$ type. The point P is a sixfold point on the nodal curve, hence the residual is a c_6 having a fourfold point at P, and four simple points on c_4. $c_{4,2}^2 + c_{6,4}^2$ (23). c_6 is unicursal. If $\lambda = \mu$, the surface becomes the developable on $c_{4,2}^2$.

12. If c_4 is of the second kind, project it from any point P upon it. The result is a K_3. Through P pass two generators to the scroll. A plane section will cut K_3 in a nodal cubic and the lines which are the projections of the generators will connect corresponding points. Let $z = 0$ be the plane of c_3; its equations are

$$x = \lambda^2 (\lambda - 1), \; y = \lambda (\lambda - 1)^2, \; w = 1.$$

If $\lambda = \dfrac{\alpha\mu + \beta}{\gamma\mu + \delta}$ be substituted in the equation of a line joining λ to μ, the result is a quartic in μ. Let x_1, y_1, w_1 be the point in which l pierces $z = 0$. There are four lines through this point which belong to the projection of the system.

Through no point of c_4 can pass more than three generators. At such a point, two of the generators belong to that point, while the third is a trisecant having the other two points for corresponding points. There are four such points on c_4, hence it follows that the residual nodal curve is a (possibly reducible) c_6 having four double points on c_4. It is unicursal. The four trisecants of c_4 which are generators to S_6 lie on the S_2 on which c_4 lies. S_6 and H_2 intersect in c_4 and these four generators. $c_4^2 + c_{6,2}^2$ (24). When in particular $\lambda = \mu$, the surface reduces to the developable on the rational quartic and the double points on the nodal sextic become cusps. The reciprocal is the sextic developable on a c_6 having four cusps and six apparent double points.

13. The scrolls having $c_{4,2}^2$ and a composite nodal sextic will first be considered. That in which $c_{6,4}^2$ becomes $2c_{3,2}^2$ was given as (9). Suppose the parameter has the values 0 and 1 at the node. If $\lambda = x$, $\mu = 0$, $\mu = x$, $\lambda = 1$ be two pairs of corresponding elements, the line joining the node to the point x is a double generator. The symbol now becomes $c_{4,2}^2 + c_{5,3}^2 + g^2$ (25). In particular, the surface may become the developable on $c_{5,3}$ and having one double generator. The $c_{5,3}$ may break up into a c_2 and $c_{3,2}$, the node of $c_{3,2}$ lying on c_2. Establish a (2, 2) correspondence between c_2, $c_{3,2}$ such that λ (of c_2) $= 0$, $\mu = 1, 0$ at the node, and intersecting each other again at $\lambda = 1$, $\mu = x$. Let (0, 0) be a double self-corresponding point and (1, 1), (1, x) be simple points. The double generator is in the plane of the conic. The symbol is $c_{4,2}^2 + c_{3,2}^2 + c_2^2 + g^2$ (26). If (x, 0), (1, x); (l, 1), (0, l) be two pairs of corresponding elements and $2x - l - xl = 0$, two double generators pass through the node and the residual is another $c_{4,2}^2$, having its node at the same point. $2c_{4,2}^2 + 2g^2$ (27). This surface is self-dual. If, in particular, $l = x = -1$, the two double generators become tacnodal and the second $c_{4,2}$ breaks up into $2c_2$ which touch. This surface, which is also self-dual, has already been given (4).

14. If the c_4 has a cusp, the resulting S_6 must have a double generator passing through it. The general cuspidal quartic may be written $x = \lambda^2$, $y = \lambda^3$, $z = \lambda^4$, $w = 1$. A line joining λ to $\dfrac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$ has the equations

$$\begin{aligned}
[\lambda^2(\gamma\lambda + \delta)^2 + (\alpha\lambda + \beta)^2]\, x - \lambda^2(\alpha\lambda + \beta)^2 w &= z(\gamma\lambda + \delta)^2, \\
[\lambda(\gamma\lambda + \delta) \quad + (\alpha\lambda + \beta)]\, y - \lambda(\alpha\lambda + \beta)\, x &= z(\gamma\lambda + \delta).
\end{aligned}$$

Two forms may exist; the general case in which the residual is $c_{5,3}^2$ (28) and that in which $c_{5,3}^2$ breaks up into $c_{3,3}^2+c_3^2$ (29). If $\beta=0$, the scroll becomes a quintic. If $\alpha=\delta=0$ and $\beta=\gamma$, c_4 is simple. The scroll has $x=0$, $z+w=0$ for double directrix, $x=0$, $y=0$ for triple generator, and $y=0$, $x+z=0$ for fourfold directrix. If $\lambda=\mu$, the ordinary quintic developable results.

Any [2] involution will also describe an S_6 of form (28). The involution may be cubical. If $\lambda=0$, $\mu=0$ is a pair of corresponding values, the surface reduces to a quintic.

Let $\lambda+\mu=\alpha$, $\lambda\mu=\beta$.
The equations of a generator are

$$(\alpha^2-\beta)x=\beta^2w+\alpha y,\quad z+\beta x=\alpha y.$$

A [2] involution is of the form

$$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0,$$

hence
$$\alpha=\frac{\phi_2(t)}{\psi_2(t)},\quad \beta=\frac{\chi_2(t)}{\psi_2(t)},$$

from which a generator has a [2, 4] type. The involution will be cubical if

$$a(a+2f)-4hg+bc=0.$$

Any plane containing two generators will, in this case, contain a third one, hence an infinite number of nodal cubics lie on the surface.

15. When c_4^2 has three apparent double points, $c_{6,3}^2$ may break up into two twisted cubics; this was (11). If a c_4 be chosen such that two trisecants cut the curve in a point having a singular osculating plane, the c_6 reduces to a c_3^2 (30). If $\lambda=\mu$, the trisecants become (ordinary) tangents and the surface is developable. (Rohn's, No. 6.)* In the general case, a trisecant may become a double generator in the same manner as for the nodal quartic. The residual is a c_5^2, having two double points (31). The limiting case $\lambda=\mu$ is possible when c_4 has

* K. Rohn, "Die rationale Raumcurve 4. Ordnung, zweiter Species." Explanatory memoir to the models of series XXI, Schilling's catalogue of mathematical models (1893).

an inflexional tangent. The scroll becomes the developable with one double generator. The residual c_5 may break up. Given a c_3 and a c_2 having three points in common. Establish a (2, 2) correspondence between them, having two simple self-corresponding points and one double element. The residual is a c_5 having two double points (32). Finally, if A on c_3 corresponds to B on c_2 and A on c_2 to B on c_3, the line AB is a double generator. The residual is c_4^2, through AB. $g^2 + c_4^2 + c_3^2 + c_2^2$ (33).

Two trisecants may be double generators. The residual is another c_4^2 of the same form as the first. This surface is self-dual, $2c_4^2 + 2g^2$ (34). If the trisecants, which count as double generators are inflexional tangents and $\lambda = \mu$, the developable with two stationary generators, is the result. This surface is still self-dual. The two forms in which c_4 breaks up into two c_2 have already been given. (2) and (7).

16. The forms of S_6 exist, having a c_4 of the first kind for nodal curve. Let S_2 be a hyperboloid containing c_4. Every generator g of S_2 cuts c_4 in two points P_1, P_2. Express a quadratic involution between g, g' of S_2, the latter cutting c_4 in P_1', P_2'. Connect each of the points P_i with each of the points P_k'. The scroll thus described will be order 8 and genus 2, but the order may be reduced to 6 by taking two tangents to c_4 for the double elements of the involution. The (2, 2) involution is elliptic and c_4 is elliptic, hence the residual nodal curve is another c_4 of the same form as the given one. Neither c_4 can break up. $2c_4^2$ (35).

§6.—S_6 *Having a Tacnodal Curve.*

17. Analogous to the surfaces having a tacnodal straight line, various forms of sextic scrolls exist having one or more tacnodal curves. The former were characterized by the fact that the plane containing two generators through a point on the double line will also contain the line itself. In the latter case the plane of two generators which pass through a point on a nodal curve must contain the tangent to the curve at that point.

Given a conic c_2 in the plane π, and having the parameter λ; also given a point P not in π. Construct the quadric cone K_2 from P to c_2. Pass a plane p through P, tangent to K_2 and in it take a conic c_2' not passing through P. Each tangent plane of K_2 will cut c_2' in two points P_1, P_2. Connect the point λ with P_1 and with P_2. The conic c_2' will cut π in two points, through each of which

passes one generator lying in π. The plane p cuts c_2 in two consecutive points Q_1, Q_2. The tangent line at Q_1 and the point P determine a plane which cuts c_2' in two points collinear with Q_1P. Hence Q_1P is a double generator and Q_2P is a double generator. c_2' is simple because apart from p only one tangent plane can be drawn to K_2 through the line connecting any point of c_2' with P. The surface is a rational S_6. The residual nodal curve is a nodal c_4. $\overline{2c_2^2} + c_{4,2}^2 + \overline{2g^3}$ (36). If a plane section through the double generator reduces to a perfect square, the $c_{4,2}$ breaks up into $2c_2^2$. $\overline{2c_2^2} + 2c_2^2 + \overline{2g^3}$ (37).

Analytically, the following procedure may be adopted: Let c_2 be defined by $x = \lambda$, $y = \lambda^2$, $z = 0$, $w = 1$, and P be (0, 0, 1, 0). The tangent to c_2 at λ and, passing through P is $\lambda^2 w - 2\lambda x + y = 0$. Let $y = 0$ be the equation of p, and let c_2' be defined by $\phi_2(x, 0, z, w) = 0$. If the tangent plane cuts c_2' in $x_1, 0, z_1, w_1$, then $y_1 = 0$, $\phi_2(x_1, 0, z_1, w_1) = 0$, $\lambda^2 w_1 - 2\lambda x_1 = 0$. These equations and those of a line joining λ to $(x_1, 0, z_1, w_1)$ are sufficient to eliminate λ, x_1, z_1, w_1. The result is the desired S_6. If c_2, c_2' have a point in common the surface will be a quintic, on which c_2' is a simple conic and c_2 a tacnodal conic. If c_2, c_2' touch each other, the surface becomes a quadric cone counted twice.

18. Now suppose p is not tangent to K_2. If c_2 c_2' intersect in two points, the same construction as before will generate a sextic, but c_2' will be double since two tangent planes to K_2 can be drawn through any line in its plane. As a (2, 2) correspondence between c_2, c_2' is defined, without any double element, the scroll will be elliptic. The residual nodal curve is a non-singular cubic. $[c_3^2] + c_2^2 + \overline{2c_2^2}$ (38). In the plane of the tacnodal conic the simple generators are tangents at the intersections of c_2 and c_2'.

19. Given a non-singular $[c_3]$ lying in $w = 0$ and cutting c_2 in one point. Lines joining corresponding points as defined in 17 will generate an S_6, $p = 1$ of symbol $c_3^2 + \overline{2c_2^2}$ (39). Let $[c_3]$ be defined by $z^2 y = x(ax^2 + bxy + cy^2)$. Since $2x_1\lambda = y_1$, from the equations of a generator we easily obtain

$$\begin{vmatrix} 2bw & f & g & ax^2 & 0 \\ 4cw & w(2bw - 8cx) & f + 4cyw & g & ax^2 \\ 1 & -2x & y & 0 & 0 \\ 0 & w & -2x & y & 0 \\ 0 & 0 & w & -2x & y \end{vmatrix} = 0$$

as the equation of the scroll; $f = 4cx^2 - 4bxw + aw^2 - 4cyw$, $g = 2bx^2 - 2axw - 2z^2$. If $[c_2]$ be not in a tangent plane to K_2, but cut c_2 in two points, the tangent at each passing through P, (38) would result. The three generators in $w = 0$ coincide with $x = 0$, $w = 0$.

20. Let c_3 be a nodal cubic in $w = 0$, so that $b^2 - 4ac = 0$. The line joining the node to the corresponding λ is a double generator. $c^2_{5,3} + 2c^2_2 + g^2$ (40). When g^2 does not lie in the plane of c_2, c_5 must have a threefold point at its intersection with c_2. When g^2 lies in the plane of c_2, it is a tangent to the latter.

21. Suppose P does not lie in the plane of c'_2, but that the latter has one point in common with c_2. In order for the surface to be a sextic, P must lie in the common tangent plane at the point of intersection of c_2, c'_2. Both conics are double and the surface is rational. The point P is so chosen that from it c_2, c'_2 project into cones having double contact. In the second common tangent plane is a g^2, and the residual nodal curve is a c^2_3. $c^2_2 + 2c^2_2 + c^2_3 + g^2$ (41). If c'_2 touch c_2, but not lie in a plane through P, the common tangent is a g^2. The residual curve is a plane nodal cubic having its node at the point of contact of c_2, c'_2. $c^2_2 + 2c^2_2 + c^2_{3,2} + g_2$ (42). This form can also be generated by establishing a similar correspondence between c_2 and $c_{3,2}$.

22. Given a c_3, $x = \lambda$, $y = \lambda^2$, $z = \lambda^3$, $w = 1$. The equation of a plane t containing the tangent to c_3 at λ is

$$\lambda^2 w - 2\lambda x + x(\lambda^2 x - 2\lambda y + z) = 0.$$

Given a conic c_2, defined by $ax + by + cz + dw = 0$, $\phi_2(x, y, z, w) = 0$, such that c_2 has three points on c_3. The plane t will cut c_2 in two points x_1, y_1, z_1, w_1. Join each such point with λ. By substituting the coordinates of any point of c_2 in t, it becomes a quadratic in λ, hence c_2 is a double conic. Let A, B, C be the points common to c_2, c_3, let the tangent plane t contain the tangent to c_2 at A, and let $t(A)$ be a perfect square in λ. A is then a double point in the (2, 2) correspondence between c_2 and c_3. B and C are simple self-corresponding points. The residual nodal curve is another conic having one point of intersection with c_2. $\overline{2c^2_3} + 2c^2_2$ (43).

If c_2 touches c_3 and has one other point on it, an S_6, having c_2 for a tacnodal curve, may be described as follows: Let P be a fixed point on c_3. Draw the tangent planes from P to c_2. Connect the point of contact with each of the remaining points of intersection with c_3. As before, c_3 will be double. $\overline{2c_2^2} + 2c_3^2$ (44). The common tangent is a torsal generator.

23. Given

$$c_1: \quad x = \lambda, \quad y = \lambda^2, \quad z = 0, \quad w = 1,$$
$$c_2: \quad x = 0, \quad y = \mu^3, \quad z = \mu, \quad w = 1.$$

The equation of a plane containing a generator is of the form

$$\mu x + \lambda z - \lambda\mu w + m(\lambda x - y + \mu z) = 0.$$

The equation of a plane containing the tangent to c_1 at λ is

$$2\lambda x - \lambda^2 w - y + \kappa z = 0.$$

When these two planes coincide, the two generators through each point λ of c_1 lie in a plane containing the tangent. The resulting equation is

$$2\lambda\mu x + (\lambda^2 + \mu^2) z - \lambda^2\mu w - \mu y = 0.$$

If a (3, 2) correspondence between λ, μ, having (∞, ∞) and $(0, 0)$ as double elements be given, the line will generate an S_6 of symbol $\overline{2c_2^2} + c_2^3$ (45). If $\lambda^3 - \mu^2 - \lambda^2 = 0$, the equation becomes

$$\begin{vmatrix} wy - z^2 & xy + yw - x^2 - z^2 & 0 \\ wx & x^2 - z^2 + yw & xy \\ wx & x^2 & x^2 + z^2 - yw \end{vmatrix} = 0.$$

The generators in the plane $z = 0$ are $w = 0$ and $x = y$.

If between λ, μ a general (2, 1) relation exists, the result is an S_6 of symbol $c_6^3 + \overline{2c_2^2}$ (46). Similarly, if between a c_3 and a c_2, with two points on c_3, be put in (2, 1) correspondence such that each point of intersection is a simple self-corresponding point, the result has the symbol $c_4^2 + \overline{2c_3^2}$ (47).

24. Given c_3, $x=\lambda$, $y=\lambda^2$, $z=\lambda^3$, $w=1$ and c_2, $x=y=\mu$, $z=\mu^2$ $w=1$, cutting c_3 in three points $\lambda=\mu=0, 1, \infty$. Write the condition that the plane containing a generator from λ will also contain the tangent to c_3 at λ. Impose the condition that the same plane contains the tangent to c_2 at μ. Finally, make $(0, 0)$ a double element and $(1, 1)$, (∞, ∞) each simple self-corresponding elements in a $(2, 2)$ correspondence between λ, μ. The resulting S_6 has the symbol $\overline{2c_3^2}+\overline{2c_3^2}$ (48).

25. Given a $c_{5,3}$. Establish a projective correspondence between two points on the curve in such a way that the two generators passing through any point of the curve lie in a plane containing the tangent to the curve at that point. Two of the values of the parameter at the triple point should be corresponding points of the projective correspondence. $\overline{2c_{5,3}^2}$ (49). If instead of a $c_{5,3}$ a rational c_5 with two double points be given, and upon it a $(1, 1)$ correspondence with one value of the parameter at each node as self-corresponding element, the resulting scroll will have this $\overline{2c_{5,3}^3}$ as tacnodal curve (50).

26. Given c_3; $x=\lambda^3$, $y=\lambda$, $z=0$, $w=1$ and $[c_3]$ defined by

$$y=0,\ xz\,(a'z+a''w)+zw\,(bx+b'z)+wx\,(cx+c'z+c''w)=0.$$

Describe a K_3 from $(0, 0, 1, 0)$ to c_3. The tangent plane $\lambda^3w+x-2\lambda y=0$ will cut $[c_3]$ in $(x_1, 0, z_1, w_1)$, $[c_3]$ will be a double curve. By letting $d=a''+b+c'$, and writing $a'z^2-dzw+cwx-cy^2+c''w^2\equiv\phi$, the equation becomes

$$\begin{vmatrix} \phi & y\,(dz-2c''w) & c''y^2-b'z^2 & 0 \\ 0 & \phi & y\,(dz-2c''w) & c''y^2-b'z^2 \\ w & -2y & x & 0 \\ 0 & w & -2y & x \end{vmatrix}=0.$$

This is the equation of (38). If $c=0$, c_3 has a node at $(1, 0, 0, 0)$ and $w=0$, $z=0$ is a g^2. The form is now (42), and its reciprocal is (37). A plane $x=\alpha w$ will have in each case a singularity of the same form at $(\alpha, -\alpha, 0, 1)$ as at $(\alpha, \alpha, 0, 1)$. If the condition be imposed that the intersection of this plane be an oscnode, the other c_3^2 of (38) will be consecutive to the given one. The tacnodal tangent of any plane section is the tangent to K_3 in that plane. The conic

will be oscnodal in each of the three forms mentioned. Their symbols become:

$$\overline{3c_3^2} + [c_3^2] \ (51), \quad \overline{3c_2^2} + c_{3,3}^3 + g^2 \ (52), \quad \overline{3c_2^2} + c_2^3 + \overline{2g^3} \ (53).$$

The last form can also be derived as a limiting case of (3).

§7.—*Corrections to Previous List.*

27. The surface of genus 3 cannot have a nodal curve of order 7, so that type I, $p = 3$, c_7^2 is impossible. (Journal, Vol. XXV, p. 267). Wiman* proved that for every scroll not contained in a linear congruence $6p \leq (n-2)(n-3)$. Hence, when $n = 6$, $p \ngtr 2$.

I assumed that two plane non-singular quartics could be found which are in (1, 1) point correspondence without being projective. Given two non-singular plane quartics c in plane π and c' in π', between which exists a (1, 1) point correspondence. Superpose the curves in such a way that the point A is self-corresponding, and that B, B', two other corresponding points, are collinear with A. This is no restriction on either curve. Now construct another curve x projective with c', such that $A = A$, B goes into B'. Between c, x exists a (1, 1) correspondence with two self-corresponding points of intersection, hence lines joining corresponding points will generate a sextic scroll of genus 3. From the preceding theorem a sextic scroll of genus 3 must belong to a linear congruence. A quartic curve can lie on such a scroll only when the residual section is a double generator. Hence the line AB is a double generator. Generators were defined as lines joining corresponding points. If AB cuts c in C, D, and x in C', D', it follows that c corresponds to c' (or D'), and that D corresponds to D' (or C'). When a (1, 1) correspondence between the points of a line exists, the correspondence must be projective. By properly choosing A, B any line in the plane can be taken for AB, hence c, x are projective. Since c', x are projective, c, c' are projective. The same reasoning can be applied to non-singular plane curves of any order, hence: *when a* (1, 1) *correspondence can be established between the points of two non-singular plane algebraic curves, the curves are projectively equivalent.*†

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* Klassifikation af regelytorna af sjette graden. Lund (Dissertation) 1892.

† This theorem was proved for cubic curves by Schwarz, "Ueber die geradlinigen Flächen fünften Grades," *Crelle*, Vol. 64, p. 27. For curves of order > 4 probably (1, 1) correspondence other than projectivity can exist only for much lower genus than the maximum.

§—8. *Table of New Results.*

1	$2c_2^2 + c_4^2 + g^2$	$p = 1$	
2	$2c_2^3 + c_4^3 + 2g^3$		3
3	$4c_2^3 + \overline{2g^2}$		2
4	$2c_2^2 + c_{4,2}^3 + \overline{2g^3}$		2
5	$2c_2^2 + c_{6,2}^3$		3
6	$2c_2^3 + c_{5,2}^2 + g^2$		3
7	$2c_2^2 + c_4^2 + 2g^2$		3
8	$4c_2^3 + 2g^2$		3
9	$2c_{3,2}^2 + c_{4,2}^2$		2
10	$2c_{3,2}^3 + 2c_2^2$		2
11	$2c_3^3 + c_4^2$		3
12	$2c_3^2 + 2c_2^3$		3
13	$c_{5,3}^2 + c_2^3 + c_{3,2}^3$		2
14	$c_{4,2}^3 + c_2^2 + c_{3,2}^2 + g^3$		2
15	$c_3^3 + c_{7,2}^2$		3
16	$c_3^3 + c_{6,2}^3 + g^2$		3
17	$c_{3,2}^2 + c_{7,4}^2$		2
18	$c_{3,2}^2 + c_{6,3}^2 + g^2$		2
19	$2c_{5,3}^2$		2
20	$2c_{5,2}^2$		3
21	c_3^3	$p = 1$	
22	$c_3^3 + g^3$		3
23	$c_{4,2}^2 + c_{6,4}^2$		2
24	$c_4^2 + c_{6,2}^2$		2
25	$c_{4,2}^3 + c_{5,3}^2 + g^3$		2
26	$c_{4,2}^3 + c_{3,2}^2 + c_2^2 + g^2$		2
27	$2c_{4,2}^3 + 2g^2$		2
28	$c_{4,\overline{2}}^3 + c_{5,2}^2 + g^3$		2
29	$c_{4,\overline{2}}^3 + c_{3,2}^2 + c_2^2$		2
30	$c_4^3 + c_2^3$		3
31	$c_4^3 + c_{5,2}^2 + g^2$		3
32	$c_3^2 + c_2^3 + c_{5,2}^2$		3
33	$c_4^2 + c_3^3 + c_2^3 + g^3$		3
34	$2c_4^2 + 2g^3$		3
35	$2c_4^3$ (first kind)	$p = 2$	
36	$\overline{2c_2^2} + c_{4,2}^2 + \overline{2g^2}$		2
37	$\overline{2c_2^2} + 2c_2^3 + \overline{2g^2}$		2
38	$[c_3^2] + c_2^2 + \overline{2c_2^2}$	$p = 1$	
39	$\overline{2c_2^2} + c_{5,2}^3$	$p = 1$	
40	$\overline{2c_2^2} + c_{5,2}^2 + g^2$		2
41	$\overline{2c_2^3} + c_3^3 + c_2^2 + g^2$		3
42	$\overline{2c_2^2} + c_{3,2}^2 + c_2^2 + g^2$		2
43	$\overline{2c_3^2} + 2c_2^2$		3
44	$\overline{2c_2^2} + 2c_3^2$		3
45	$\overline{2c_2^3} + c_2^3$		3
46	$\overline{2c_2^2} + c_{6,4}^3$		2
47	$\overline{2c_3^2} + c_4^3$		3
48	$\overline{2c_2^2} + \overline{2c_3^3}$		3
49	$\overline{2c_{5,3}}$		2
50	$\overline{2c_{5,2}}$		3
51	$\overline{3c_2^2} + [c_3^3]$	$p = 1$	
52	$\overline{3c_2^2} + c_{3,2}^2 + g^3$		2
53	$\overline{3c_2^2} + c_2^3 + \overline{2g^3}$		3

CORNELL UNIVERSITY, *July 26, 1904.*

Determination of the Ternary Modular Groups.

By Leonard Eugene Dickson.*

1. The determination of all groups of linear homogeneous transformations on m variables with coefficients in the $GF[p^n]$ falls naturally into two cases: (i) order a multiple of p; (ii) order prime to p. In the second case, the canonical form of any transformation merely multiplies each variable by a constant, and the problem is analogous to that of the determination of the finite groups of collineations in m variables.† This separation of cases was followed in the treatment of binary groups.‡

In his elaborate memoir on ternary groups, Burnside ‖ makes the limitation that p^2+p+1 shall be the product of at most two prime factors >3 or else the triple of such a product. His discussion is occasionally incorrect. In particular, he misses** the groups with an invariant ternary quadratic form.

The present paper on the ternary groups of order a multiple of p employs methods entirely different from those used by Burnside. There is no limitation on the odd prime p. Moreover, a representative of each set of conjugate subgroups is exhibited in explict form.

2. The order of the group G of all ternary transformations modulo p of determinant unity is $p^3(p^3-1)(p^2-1)$. Every subgroup of order a power of

* Research Assistant to the Carnegie Institution of Washington.

† References to the work of Klein, Gordan, Jordan, and Valentiner are given in the new attack by Blichfeldt, Transactions, Vol. 4, p. 387; Vol. 5, p. 310.

‡ Compare the related problem of the unary linear fractional group treated by Moore, Burnside, Wiman, and Dickson (references in Linear Groups, p. 260). The writer has recently made a complete determination of the binary groups of determinant unity in the $GF[p^n]$.

‖ Proc. Lond. Math. Soc., Vol. XXVI, pp. 58-106.

§ Ibid., pp. 77, 81, 102-104. Cf. Amer. Journ. Math., Vol. XXII (1900), p. 231.

** Burnside, ibid., p. 81.

p is, therefore, conjugate with a subgroup of the group G_{p^3} of the operators

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}. \tag{1}$$

In case a subgroup of G_{p^3} is defined by certain independent relations $r_1 = 0, \ldots, r_s = 0$ between a, b, c, we denote it $\{r_1 = 0, \ldots, r_s = 0\}$. We employ also the usual notation

$$B_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda \xi_j, \quad \xi_k' = \xi_k, \qquad (k \neq i). \tag{2}$$

Since the commutator subgroup of G_{p^3} is formed of the operators $B_{3,1,s}$, and since the p^{th} power of (1) is of the form $B_{3,1,s}$, it follows that G_{p^3} has exactly $p+1$ subgroups of order p^2. But any linear relation between a and c defines such a subgroup. *Hence* the subgroups of order p^2 of G_{p^3} are $\{a = 0\}$ and $\{c = ta\}$, $t = 0, 1, \ldots, p-1$.*

The $p+1$ subgroups C_p of $\{a = 0\}$ are $\{a = c = 0\}$, $\{a = 0,\ b = wc\}$, $w = 0, 1, \ldots, p-1$. Now $B_{2,1,w}$ transforms the latter into $\{a = b = 0\}$. The $p+1$ subgroups C_p of $\{c = ta\}$ are $\{a = c = 0\}$, $\{c = ta,\ b = \frac{1}{2}ta^2 + va\}$. When the latter is transformed by $B_{3,2,s}$, the only change is the replacement of v by $v + s$. We may thus make $v = 0$. *Within G every subgroup of order p is conjugate with $(B_{3,2,1})$ or $J_t \equiv \{c = ta,\ b = \frac{1}{2}ta^2\}$, $t \neq 0$.*

3. The conditions for $\{c = ta\}(\alpha_{ij}) = (\alpha_{ij})\{C = sA\}$ are

$$\alpha_{12} = \alpha_{13} = s\alpha_{23} = t\alpha_{13} = 0, \quad ta\alpha_{33} = sA\alpha_{22}, \tag{3}$$

$$a\alpha_{22} + b\alpha_{32} = A\alpha_{11}, \quad a\alpha_{32} + b\alpha_{33} = B\alpha_{11} + sA\alpha_{21}. \tag{4}$$

Since $|\alpha_{ij}| \neq 0$, t and s are both zero or both $\neq 0$. For $t = s = 0$, the conditions reduce to $\alpha_{12} = \alpha_{13} = 0$, since (4) serve to determine A and B in terms of a and b, or vice versa. For $t \neq 0$, $s \neq 0$, then $\alpha_{23} = 0$, $A = a\alpha_{22}\alpha_{11}^{-1}$ by $(4)_1$, and $t\alpha_{33} = s\alpha_{22}^2\alpha_{11}^{-1}$ by the final condition (3). Now $|\alpha_{ij}| = \alpha_{11}\alpha_{22}\alpha_{33} = 1$. Hence $t = s\alpha_{22}^3$. Let d be the greatest common divisor of 3 and $p-1$. If $d = 1$, every integer is a cubic reside modulo p, so that $t = s\alpha_{22}^3$ can always be satisfied. If $d = 3$, the two groups are conjugate if, and only if, t/s is a root of $y^{\frac{1}{3}(p-1)} \equiv 1 \pmod{p}$. *If $p = 3$ or if $p = 3l - 1$, the groups $\{c = ta\}$, $t = 1, \ldots, p-1$, are all conjugate within G; if $p = 3l + 1$, they fall into three sets represented by $t = 1, \beta, \beta^2$,*

* Cf. Bulletin, Vol. X (1904), p. 392, formula (9).

where β is a particular non-cubic residue of p. For any p, $\{c=ta\}, t\neq 0$, is commutative with only the operators $(5)_1$, with $a_{23}^3=1$; $\{c=0\}$ with only $(5)_2$; $\{a=0\}$ with only $(5)_3$; G_{p^3} with only $(5)_1$, the determinant to be unity in each case:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \qquad (5)$$

By a similar proof, *there are exactly d non-conjugate sets of cyclic $J_t \equiv \{c=ta,\, b=\frac{1}{2}ta^2\}$, $t\neq 0$; J_t is commutative with only the operators* $(5)_1$ *with $a_{22}^2=a_{11}a_{33}$, $a_{22}^3=1$, $a_{32}=ta_{33}a_{21}a_{22}^{-1}$. Also, $(B_{3,2,1})$ is commutative with only the operators* $(5)_3$ *with $a_{21}=0$, $a_{11}a_{22}a_{33}=1$.*

4. Lemma. *The only factors $\equiv 1 \pmod{p^2}$ of $(p^3-1)(p^2-1)=\omega$ are $1, \omega$.*

Let $\omega=(1+p^2x)q$, $x>0$. Then $q\equiv 1 \pmod{p^2}$, $q=1+p^2y$, $y\geqq 0$. Then $p^3-p-1=x+y+p^2xy$. Hence

$$x+y=tp^2-p-1, \quad xy=p-t, \qquad t>0.$$

Now $y=0$ gives $1+p^2x=\omega$. Next, for $y\geqq 1$, $x\geqq 1$, the second condition requires that x and y be each $<p$. By the first, $tp^2-p-1\leqq 2p-2$. But $p^2-3p+1>0$ if $p\geqq 3$. For $p=2$, the lemma is evidently true.

5. Lemma. *Any binary transformation $B=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$, $\gamma\neq 0$, and all the $E_\lambda=\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$ generate every binary transformation T of determinant unity.*

Indeed,

$$E_{-\alpha\gamma^{-1}}BE_{-\delta\gamma^{-1}}=\begin{pmatrix}0 & \gamma\\ \tau & 0\end{pmatrix}, \quad \tau=-\gamma^{-1}(\alpha\delta-\beta\gamma)\neq 0.$$

The latter transforms E_λ into $F_\sigma=\begin{pmatrix}1 & \sigma\\ 0 & 1\end{pmatrix}$, where $\sigma=\lambda\gamma\tau^{-1}$ may be made arbitrary. But the E_λ and F_σ are known to generate every T.

6. Let H be a subgroup of order p^3N, normalized to contain G_{p^3}. If the latter is self-conjugate, the operators of H are (§3) all of the form $(5)_1$, so that H is given by the extension of G_{p^3} by certain operators

$$M_{\alpha,\beta,\gamma}:\quad \xi_1'=\alpha\xi_1, \quad \xi_2'=\beta\xi_2, \quad \xi_3'=\gamma\xi_3, \quad \alpha\beta\gamma\equiv 1.$$

Next, let H contain $k > 1$ groups conjugate with G_{p^3}. Unless $H = G$, $k \not\equiv 1 \pmod{p^2}$ by §4. Hence* G_{p^3} and one of its conjugates under H have a common subgroup of order p^2. Hence† H has an operator S commutative with this G_{p^2} but not with G_{p^3}.

(i). Let first G_{p^3} be $\{a = 0\}$. Then S is of the form $(5)_3$ with $\alpha_{12} \not\equiv 0$. By choice of α and β, $B_{3,1,\alpha} B_{3,2,\beta}\, S$ has $\alpha_{31} = \alpha_{32} = 0$, $\alpha_{12} \not\equiv 0$. Hence (§5) H contains every binary transformation B of determinant unity on ξ_1, ξ_2. *If H contains an operator $\Sigma = (\beta_{ij})$, β_{13} and β_{23} not both zero, then $H = G$.* Indeed, applying $\xi_1' = \xi_2$, $\xi_2' = -\xi_1$ on the right of Σ if necessary, we may set $\beta_{13} \not\equiv 0$. Applying $M_{\beta_{13}^{-1}, \beta_{13}, 1}$ on the right, we reach Σ_1 with $\beta_{13} = 1$. Then

$$\Sigma_1\, B_{2,1,-\beta_{23}}\, B_{3,1,-\beta_{33}} = (\gamma_{ij}), \quad \gamma_{13} = 1, \; \gamma_{23} = \gamma_{33} = 0, \; \begin{vmatrix} \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{vmatrix} = 1.$$

Multiplying on the right by the inverse of $\begin{pmatrix} \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$ on ξ_1 and ξ_2, and then on the left by $B_{2,1,-\gamma_{11}}\, B_{3,2,-\gamma_{12}}$, we obtain $(\xi_1\, \xi_3\, \xi_2)$. This transforms $B_{3,2,\rho}$ and $B_{1,2,\rho}$ into $B_{1,3,\rho}$ and $B_{2,3,\rho}$, respectively. But all the $B_{i,j,\rho}$ generate G.

(ii). Let next G_{p^3} be $\{c = 0\}$. Then S is of the form $(5)_2$ with $\alpha_{23} \not\equiv 0$. By choice of a and b, $SB_{2,1,a}B_{3,1,b}$, has $\alpha_{21} = \alpha_{31} = 0, \alpha_{23} \not\equiv 0$. Then (§5), H contains every binary transformation on ξ_2, ξ_3 of determinant 1. If H contains an operator not of the form $(5)_2$, then $H = G$; the proof is quite similar to that in case (i).

(iii). Let finally G_{p^3} be $\{c = ta\}$, $t \not\equiv 0$. Every operator commutative with it is of the form $(5)_1$ and hence is commutative with G_{p^3}.

THEOREM.—*Within G every subgroup of order a multiple of p^3 is conjugate with one of the following:* (i) *the group of all the p^3fg operators* $(5)_1$ *with* $\alpha_{11}^f = 1$, $\alpha_{22}^g = 1$, $\alpha_{33} = \alpha_{11}^{-1}\alpha_{22}^{-1}$; (ii) *the group of all the $fp^3\,(p^2 - 1)$ operators* $(5)_2$ *with* $\alpha_{11}^f = 1$, $\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = \alpha_{11}^{-1}$; (iii) *the group of all the $fp^3\,(p^2 - 1)$ operators* $(5)_3$ *with* $\alpha_{33}^f = 1$, $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \alpha_{33}^{-1}$. *Here f and g may be any divisors of $p - 1$.*

7. Let H be a subgroup of order p^2N, normalized to contain a subgroup of order p^2 of G_{p^3}. We prove that this G_{p^2} must be self-conjugate under H. If the number of conjugates to G_{p^2} is ω, H is of index p under G, whereas the order

*Cf. Burnside's Theory of Groups, p. 94, Cor. II. †Ibid., p. 97.

of the simple linear fractional group $LF(3, p)$ does not divide $p!$. Hence (§4) G_{p^2} and one of its conjugates under H have a common cyclic C_p, and H has an operator commutative with C_p but not with G_{p^2}. By §3, this is impossible if C_p is J_t or $(B_{3,2,1})$, since, in the latter case, G_{p^2} must be $\{a=0\}$.

The quotient of the group of the operators $(5)_2$ by $\{c=0\}$ may be taken concretely as the group of the operators $(5)_2$ with $\alpha_{21}=\alpha_{31}=0$. We must take a group of the latter operators of period prime to p. The corresponding group of binary operators of determinant 1 on ξ_2 and ξ_3 must have the order 1, 2, $4k$, 24, 48 or 120 (see third foot-note to § 1).

The quotient of the group of the operators $(5)_1$ with $\alpha_{22}^3=1$ by $\{c=ta\}$ may be taken concretely as the group Q of the products $R_{\alpha,\rho}M_\epsilon$,

$$R_{\alpha,\rho}=\begin{pmatrix}\rho^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & \alpha & \rho\end{pmatrix},\quad M_\epsilon=\begin{pmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{pmatrix},\quad \varepsilon^d=1.$$

Within Q any subgroup of order prime to p is conjugate a group of operators $R_{0,\rho}M_\epsilon$.

THEOREM.—*Every subgroup H of order a multiple of p^2 but not of p^3 contains a self-conjugate G_{p^2}. Within G, H is conjugate with a group of operators $(5)_1$ with*

$$\alpha_{22}^3=1,\quad \alpha_{11}\alpha_{22}\alpha_{33}=1,\quad \alpha_{32}=t\alpha_{21}\alpha_{33}\alpha_{22}^{-1},$$

where t is a constant having one of d values; or a group of operators $(5)_2$ in which the $\begin{pmatrix}\alpha_{22} & \alpha_{23}\\ \alpha_{32} & \alpha_{33}\end{pmatrix}$ define a binary group of order prime to p; or a group of operators $(5)_3$ with an analogous restriction.

8. Let H be a subgroup of order pN, normalized to contain $(B_{3,2,1})$. Suppose, first, that the latter is self-conjugate. The quotient-group Q of the group of operators $(5)_3$ with $\alpha_{21}=0$ by $(B_{3,2,1})$ may be taken concretely as the group of the $p^2(p-1)^2$ operators $(5)_3$ with $\alpha_{21}=\alpha_{31}=0$, $\alpha_{11}\alpha_{22}\alpha_{33}=1$. Q contains self-conjugately the group of the p^2 operators $(6)_1$:

$$\begin{pmatrix}1 & \beta & 0\\ 0 & 1 & 0\\ 0 & \gamma & 1\end{pmatrix},\quad \begin{pmatrix}\alpha_{11} & \beta(\alpha_{22}-\alpha_{11}) & 0\\ 0 & \alpha_{22} & 0\\ 0 & \gamma(\alpha_{22}-\alpha_{33}) & \alpha_{33}\end{pmatrix}. \tag{6}$$

Now $(6)_1$ transforms $M_{a_{11}, a_{22}, a_{33}}$ into $(6)_2$. Hence Q contains p^3 subgroups of order $(p-1)^2$, no two of which have a common operator other than $M_{\epsilon, \epsilon, \epsilon}$, $\epsilon^d = 1$. The remaining dp^3 operators in Q are products of the form $(6)_1 \, M_{\epsilon, \epsilon, \epsilon}$. Hence, every subgroup of order prime to p of Q is conjugate within Q with a group of operators $M_{\alpha, \beta, \gamma}$. Hence H is conjugate with the first group of the theorem below.

Let next $(B_{3,2,1})$ be not self-conjugate in H, which, therefore, contains an operator $S = (\alpha_{ij})$ with $\alpha_{13}, \alpha_{23}, \alpha_{21}$ not all zero (§3). We simplify S by transforming it by operators M, $B_{1,2,\rho}$ and $B_{3,1,\rho}$, each commutative with $(B_{3,2,1})$, and by multiplying it on the right or left by $B_{3,2,\rho}$. Now, any (α_{ij}) transforms $B_{3,2,1}$ into

$$\xi_i' = \xi_i + \alpha_{i3}\eta, \qquad (i = 1, 2, 3), \tag{7}$$

where η is the function by which $(\alpha_{ij})^{-1}$ replaces ξ_2.

(i). Let $\alpha_{23} \neq 0$. Transforming S by $M_{\alpha_{23}^{-1}, 1, \alpha_{23}}$, we may set $\alpha_{23} = 1$. Transforming by $B_{1,2,-\alpha_{13}}$, we have $\alpha_{13} = 0$. Then $SB_{3,2,-\alpha_{33}}$ has $\alpha_{23} = 1$, $\alpha_{13} = \alpha_{33} = 0$. Then (7) becomes

$$\xi_1' = \xi_1, \quad \xi_2' = \alpha_{31}\xi_1 + \xi_2 - \alpha_{11}\xi_3, \quad \xi_3' = \xi_3, \qquad (\alpha_{11}, \alpha_{31} \text{ not both } 0). \tag{8}$$

If $\alpha_{11} = 0$, we reach $B_{2,1,1}$, whereas the order of H is not divisible by p^3. Hence $\alpha_{11} \neq 0$, and $B_{3,1,-\alpha_{31}\alpha_{11}^{-1}}$ transforms (8) into $B_{2,3,-\alpha_{11}}$. Hence H contains all binary transformations B of determinant unity on ξ_2, ξ_3.

(ii). Let $\alpha_{23} = 0$, $\alpha_{13} \neq 0$. Transforming by $M_{\alpha, \beta, \gamma}$ and $B_{3,1,\rho}$, we may set $\alpha_{13} = 1$, $\alpha_{33} = 0$. Then (7) becomes

$$S_1: \quad \xi_1' = \xi_1 - \alpha_{31}\xi_2 + \alpha_{21}\xi_3, \quad \xi_2' = \xi_2, \quad \xi_3' = \xi_3.$$

For $\alpha_{21} = 0$, S_1 and $B_{3,2,1}$ generate a G_{p^2}. For $\alpha_{21} \neq 0$,

$$S_1^{-1} \, B_{3,2,\rho}^{-1} \, S_1 \, B_{3,2,\rho} = B_{1,2,-\rho\alpha_{21}}.$$

But this and S_1 generate a G_{p^3}.

(iii). Let $\alpha_{23} = \alpha_{13} = 0$, $\alpha_{21} \neq 0$, whence $\alpha_{33} \neq 0$. Then (7) becomes

$$\xi_1' = \xi_1, \; \xi_2' = \xi_2, \; \xi_3' = -\alpha_{21}\alpha_{33}^2 \xi_1 + \alpha_{11}\alpha_{33}^2 \xi_2 + \xi_3.$$

But this and $B_{3,2,1}$ generate a G_{p^2}.

It remains only to discuss the groups of case (i). Suppose that H contains (β_{ij}) with β_{12}, β_{13}, β_{21}, β_{31} not all zero.

If $\beta_{12}=\beta_{13}=0$, we apply a B on the right and make also $\beta_{23}=\beta_{32}=0$, $\beta_{33}=1$, whence $\beta_{22}=\beta_{11}^{-1}$. If $\beta_{21}=0$, so that $\beta_{31}\neq 0$,

$$(\beta_{ij})^{-1}\, M_{1,-1,-1}^{-1}\, (\beta_{ij})\, M_{1,-1,-1} = B_{3,1,-2\beta_{31}\beta_{11}^{-1}},$$

and H contains a G_{p^2}. The case $\beta_{21}\neq 0$ is excluded by (iii).

Hence β_{12} and β_{13} are not both zero. Applying a B on the left, we may set $\beta_{12}=1$, $\beta_{13}=0$. Applying next a B on the right, we may set also $\beta_{23}=0$, $\beta_{33}=1$. The case $\beta_{21}\neq 0$ is excluded by (iii). Hence $\beta_{21}=0$, $\beta_{22}=\beta_{11}^{-1}$. Applying $B_{3,2,-\beta_{32}}$ on the left, we may set also $\beta_{32}=0$. Then

$$(\beta_{ij})^{-1}\, M_{1,\frac{1}{2},2}^{-1}\, (\beta_{ij})\, M_{1,\frac{1}{2},2} = \begin{pmatrix} 1 & \beta_{11} & 0 \\ 0 & 1 & 0 \\ \beta_{11}^{-1}\beta_{31} & -\beta_{31} & 1 \end{pmatrix}$$

is of period p and is commutative with $B_{3,2,1}$, so that the two generate a G_{p^2}. We have now proved the

THEOREM.—*According as a subgroup of order a multiple of p but not of p^2 contains $(B_{3,2,1})$ self-conjugately or not, it is conjugate within G with a group of pg products $B_{3,2,\rho}\, M_{\alpha,\beta,\alpha^{-1}\beta^{-1}}$, g a divisor of $(p-1)^2$, or with the group of order $pf(p^2-1)$ given by the extension of the binary group of determinant unity on ξ_2, ξ_3 by $M_{\alpha,\alpha^{-1},1}$, $\alpha^f\equiv 1$ (mod p.)*

9. Finally, let H be a subgroup of order pN, normalized within G to contain J_t. By section 3, any operator T commutative with J_t may be expressed as the product of $M_{\epsilon,\epsilon,\epsilon}$, $\epsilon^d=1$, by

$$\begin{pmatrix} \rho^{-1} & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & t\alpha\rho & \rho \end{pmatrix} = \begin{pmatrix} \rho^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ \beta-\frac{1}{2}t\alpha^2\rho & 0 & \rho \end{pmatrix} S_{\alpha\rho},\quad S_\alpha \equiv \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \frac{1}{2}t\alpha^2 & t\alpha & 1 \end{pmatrix},$$

where S_α is the general operator of J_t and $S_\alpha = S_1^{\alpha}$. The quotient group $Q=(T)\,/\,J_t$ may be taken concretely as the group of the $dp\,(p-1)$ products $M_{\epsilon,\epsilon,\epsilon}\, M_{\rho^{-1},1,\rho}\, B_{3,1,\gamma}$. Hence Q contains p groups conjugate with $(M_{\epsilon\rho^{-1},\epsilon,\epsilon\rho})$ ρ being a primitive root of p, no two of them have common operators other than $M_\epsilon \equiv M_{\epsilon,\epsilon,\epsilon}$. The remaining dp operators of Q lie in $(M_\epsilon,\, B_{3,1,\gamma})$. Within Q every subgroup of order prime to p is, therefore, conjugate with a subgroup of $(M_\epsilon,\, M_{\rho^{-1},1,\rho})$.

Let first J_t be self-conjugate in H. Then H is conjugate with a group of efp products $M_\epsilon\, M_{\sigma^{-1},1,\sigma}\, B_{3,1,\lambda}$.

Let next J_t be not self-conjugate in H. It suffices to consider the case $t=1$, since $M_{1,1,t^{-1}}$ transforms J_t into J_1, and H into a subgroup of G; the final list of the groups with J_1 must be transformed by the inverse $M_{1,1,t}$. Hence let H contain J_1 and an operator $S=(\alpha_{ij})$ with α_{12}, α_{13}, α_{23} not all zero (so that S shall not transform J_1 into a subgroup of G_{p^3}).

(i) Let $\alpha_{13}\neq 0$. Multiplying S on the left by an S_a, we may set $\alpha_{12}=0$. If then $\alpha_{32}=0$, S^{-1} has $\alpha'_{13}=0$, $\alpha'_{12}=\alpha_{13}\,\alpha_{32}\neq 0$, case (ii). Let now $\alpha_{32}\neq 0$. Then SS_a has $\alpha_{12}=\alpha_{32}=0$. Transforming by $B_{3,1,\sigma}$, which is commutative with S_a, we reach a transformation Σ with $\alpha_{12}=\alpha_{32}=\alpha_{33}=0$, $-\alpha_{13}\,\alpha_{31}\,\alpha_{22}=1$. The transform of S_a by Σ is

$$\begin{pmatrix} 1+a\alpha_{23}\alpha_{13}\alpha_{31} & -a\alpha_{13}^2\alpha_{31} & -a\alpha_{13}\left(D+\tfrac{1}{2}\,a\alpha_{22}\alpha_{13}\right) \\ a\alpha_{23}^2\alpha_{31} & 1-a\alpha_{23}\alpha_{31}\alpha_{13} & -aD\alpha_{23}-a\alpha_{23}^2\alpha_{13}-\tfrac{1}{2}\,a^2\alpha_{13}\alpha_{22}\alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

where $D=\alpha_{11}\alpha_{23}-\alpha_{13}\alpha_{21}$. If $D\neq 0$, we can determine a to make $\alpha'_{12}\neq 0$, $\alpha'_{13}=0$, case (ii). If $D=0$, the transform of Σ^{-1} by $B_{3,1,\alpha_{11}\alpha_{13}^{-1}}$ is of the form Σ with $D'=-\alpha_{23}$; then $D'=0$ requires $\alpha_{23}=\alpha_{21}=0$. In the latter case, (9) becomes W_a if we take $a=-\alpha_{23}^{-2}\alpha_{13}^{-1}$, and set $\alpha=-\alpha_{22}^{-3}$:

$$W_a=\begin{pmatrix} 1 & \alpha & \tfrac{1}{2}\alpha \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\quad V=\begin{pmatrix} 0 & 0 & \tfrac{1}{2}\alpha \\ 2\alpha^{-1}(\alpha-1) & -1 & 1-\alpha \\ 2\alpha^{-1} & 0 & 0 \end{pmatrix}.$$

Applying the method by which S was reduced to Σ, we compute $S_{-2}W_aS_{-2}$ and transform it by $B_{3,1,\sigma}$, where $\sigma=2\,(1-\alpha)\,\alpha^{-1}$. There results V, which is of the form Σ with $D=1-\alpha$. We are thus led to case (ii) unless $\alpha=1$. For $\alpha=1$, V becomes T, which transforms S_a into E_a:

$$T=\begin{pmatrix} 0 & 0 & \tfrac{1}{2} \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix},\quad E_a=\begin{pmatrix} 1 & -\tfrac{1}{2}a & \tfrac{1}{8}a^2 \\ 0 & 1 & -\tfrac{1}{2}a \\ 0 & 0 & 1 \end{pmatrix}.$$

For brevity set $N_\rho=M_{\rho,1,\rho^{-1}}$. We have the relations

$$E_a=E_1^a,\quad S_a=S_1^a,\quad S_aN_\rho=N_\rho S_{a\rho^{-1}},\quad E_aN_\rho=N_\rho E_{a\rho}, \tag{10}$$

$$T^2=\text{identity},\quad TS_a=E_aT,\quad TN_\rho=N_{\rho^{-1}}T, \tag{11}$$

$$E_cS_b=N_{a^2c^{-2}}\,S_{-bca^{-1}}E_{-a},\qquad \left(a=\frac{4c}{bc-4},\ c\neq 0,\ b\neq\frac{4}{c}\right), \tag{12}$$

$$E_cS_{4c^{-1}}=N_{4c^{-2}}S_{-c}\,T,\qquad (c\neq 0). \tag{13}$$

Every operator of the group K generated by S_1 and T can be expressed in one of the two forms

$$N_{k^2}S_bE_a = \begin{pmatrix} k^2\left(1-\frac{ab}{4}\right)^2 & -\frac{a}{2}\left(1-\frac{ab}{4}\right) & \frac{1}{8}a^2k^{-2} \\ bk^2\left(1-\frac{ab}{4}\right) & 1-\frac{ab}{2} & -\frac{1}{2}ak^{-2} \\ \frac{1}{2}b^2k^2 & b & k^{-2} \end{pmatrix}, \tag{14}$$

$$N_{k^2}S_bT = \begin{pmatrix} \frac{1}{4}b^2k^2 & \frac{1}{2}b & \frac{1}{2}k^{-2} \\ -bk^2 & -1 & 0 \\ 2k^2 & 0 & 0 \end{pmatrix}. \tag{15}$$

First, S_1 times either reduces at once to one of these forms by (10). Next, by (11),

$$T\,.\,N_{k^2}S_bE_a = N_{k^{-2}}E_bS_aT, \quad T\,.\,N_{k^2}S_bT = N_{k^{-2}}E_b.$$

For $a = 0$, $N_{k^{-2}}E_bT = N_{k^{-2}}N_{4b^{-2}}S_{-b}E_{-4b^{-1}}$, of the form (14), the equality following from (13) upon transforming its members by T. For $a \neq 0$,

$$\begin{aligned} N_{k^{-2}}E_bS_aT &= N_{k^{-2}}E_b\,.\,N_{a^2/4}E_{-a}S_{-4a^{-1}}, \text{ by (13)}, \\ &= N_{k^{-2}a^2/4}E_{-a+ba^2/4}S_{-4a^{-1}}, \quad \text{by } (10)_4, \end{aligned}$$

and hence is of one of the forms (14), (15), in view of (12) and (13). Hence the group K is composed of the $\frac{1}{2}p(p^2-1)$ distinct operators (14) and (15). These may be combined into the simple form, with the invariant $\xi_2^2 - 2\xi_1\xi_3$:

$$\begin{pmatrix} \alpha^2 & \alpha\beta & \frac{1}{2}\beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & \beta\delta \\ 2\gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \tag{16}$$

Indeed, (14) is obtained by setting

$$\alpha = k\left(1-\frac{ab}{4}\right), \quad \beta = -\tfrac{1}{2}ak^{-1}, \quad \gamma = \tfrac{1}{2}bk, \quad \gamma = k^{-1};$$

while (15) is obtained by setting $\alpha = -\frac{1}{2}bk$, $\beta = -k^{-1}$, $\lambda = k$, $\delta = 0$. Now α, β, λ, δ and their negatives give the same operator (16). Further, there are exactly $p(p^2-1)$ sets of solutions of $\alpha\delta - \beta\gamma \equiv 1 \pmod{p}$. *Hence K is simply isomorphic with the group Γ of all unary linear fractional substitutions of determinant unity modulo p.**

* Since S_1 is not conjugate with $B_{2,2,1}$, there follows the known theorem that Γ is not representable as a binary homogeneous group of determinant 1.

If ν be a particular not-square, N_ν extends K to a group K' of order $p(p^2-1)$, composed of all ternary transformations of determinant unity leaving $\xi_2^2 - 2\xi_1\xi_3$ absolutely invariant. Indeed, N_ν transforms S_1 and T into $S_{\nu-1}$ and $N_{\nu-1}T$, respectively.

Consider a group H' which contains K' and a further operator $S=(\alpha_{ij})$. Now S, TS, ST, TST do not all have $\alpha_{13}=0$. We, therefore, assume that $\alpha_{13}\neq 0$ in S. By choice of b,ρ,a, $S_bSN_\rho S_a = R$ has $\alpha_{12}=\alpha_{23}=0$, $\alpha_{13}=1$. Then

$$\Sigma \equiv R^{-1}N_{-1}RN_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha_{21}\alpha_{22}\alpha_{33} & 1-2\alpha_{21}\alpha_{32} & 2\alpha_{21}\alpha_{22} \\ 2\alpha_{21}\alpha_{32}\alpha_{33} & 2\alpha_{31}\alpha_{32}-2\alpha_{11}\alpha_{32}\alpha_{33} & 1-2\alpha_{21}\alpha_{32} \end{pmatrix}.$$

If $\alpha_{21}\alpha_{22}\neq 0$, a suitable product $S_bN_\sigma\Sigma S_a$ gives V:

$$V=\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 0 & 1 \\ \delta & -1 & 0 \end{pmatrix}, \quad V^{-1}N_\rho^{-1}VN_\rho = \begin{pmatrix} 1 & 0 & 0 \\ \gamma(\rho^{-1}-\rho) & \rho & 0 \\ \delta(\rho^{-2}-\rho^{-1}) & 0 & \rho^{-1} \end{pmatrix}.$$

We may take $\rho^3\neq 1$. Then (§3) the latter operator transforms J_1 into another subgroup of G_{p^3}, so that H' would be of order a multiple of p^2. The same is true for Σ if $\alpha_{22}=0$. If $\alpha_{21}=0$, $\Sigma = B_{3,1,k}$, $k=2\alpha_{31}\alpha_{32}-2\alpha_{11}\alpha_{32}\alpha_{33}$. If $k\neq 0$, we obtain a G_{p^2}. Hence $k=0$. Now $\alpha_{11}\alpha_{22}\alpha_{33}-\alpha_{31}\alpha_{22}=1=|S|$. Hence $\alpha_{32}=0$, and

$$S=\begin{pmatrix} \alpha_{11} & 0 & 1 \\ 0 & \alpha_{22} & 0 \\ \alpha_{31} & 0 & \alpha_{33} \end{pmatrix}, \quad TST=\begin{pmatrix} \alpha_{33} & 0 & \frac{1}{4}\alpha_{31} \\ 0 & \alpha_{22} & 0 \\ 4 & 0 & \alpha_{11} \end{pmatrix}, \quad \alpha_{22}(\alpha_{11}\alpha_{33}-\alpha_{31})=1.$$

If $\alpha_{31}=0$, $N_\sigma TST$ can be given the form W with $\delta\neq 0$:

$$W=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ \delta & 0 & \alpha^{-1} \end{pmatrix}, \quad Z=\begin{pmatrix} \alpha'_{11} & 0 & \alpha'_{13} \\ 0 & 1 & 0 \\ -4\alpha'_{13} & 0 & \alpha'_{33} \end{pmatrix}.$$

Then $N_\rho^{-1}W^{-1}N_\rho W = B_{3,1,l}$, $l=\delta(1-\rho^{-2})$. Hence $\alpha_{31}\neq 0$. Then $S^{-1}N_\rho TST$ gives Z, where

$$\alpha'_{11}=\alpha_{22}(\rho\alpha_{33}^2-\tfrac{1}{4}\rho^{-1}\alpha_{31}^2), \quad \alpha'_{13}=\tfrac{1}{4}\rho^{-1}\alpha_{11}\alpha_{22}\alpha_{31}-\rho\alpha_{22}\alpha_{33}.$$

If $\alpha_{33}\neq 0$, we take $\rho=\frac{1}{2}\alpha_{31}\alpha_{33}^{-1}$, whence $\alpha'_{11}=0$, $\alpha'_{13}=\frac{1}{4}$. Then TZ is of the form W with $\alpha=-1$ and hence transforms J_1 into another subgroup of G_{p^2}.

Hence $\alpha_{33} = 0$. If $\alpha_{11} \neq 0$, $N_\rho TST$ is of the form S with $\alpha_{31} \neq 0$, $\alpha_{33} \neq 0$, previously excluded. Hence $\alpha_{11} = \alpha_{33} = 0$ in S. Then

$$STN_2 = M\alpha_{31},\ \alpha_{31}^{-1},\ 1\,.$$

If $\alpha_{31}^3 \neq 1$, this transforms J_1 into another subgroup of G_{p^3}. We thus reach the extender $M_{\epsilon^2, \epsilon, 1} = M_{\epsilon, \epsilon, \epsilon} M_{\epsilon, 1, \epsilon^2}$, $\epsilon^3 = 1$. Now $M_{\epsilon, \epsilon, \epsilon}$ occurs in K' if, and only if, $\epsilon = 1$. Hence H' is of order $dp(p^3 - 1)$ and leaves $\xi_2^2 - 2\xi_1\xi_3$ relatively invariant.

We now pass from case (i) to the study of the group H with an operator $S = (\alpha_{ij})$, $\alpha_{13} = 0$, α_{12} and α_{23} not both zero. These properties of S are not altered when we make the normalization as at the beginning of the section, in view of which the largest subgroup G_{pwe} of H commutative with J_1 is composed of the products $M_{\epsilon, \epsilon, \epsilon} N_\rho S_a$, $\rho^w = 1$, $\epsilon^e = 1$, $\epsilon = 1$ or d. We defer to §10 the case $w = 1$, assuming $w > 1$ here.

(ii) $\alpha_{13} = 0$, $\alpha_{12} \neq 0$. Transforming H by $N_{\alpha_{12}}^{-1}$, we may set $\alpha_{12} = 1$ in S. Replacing S by a suitable product $S_a S S_b$, we may set $\alpha_{11} = \alpha_{13} = \alpha_{22} = 0$, $\alpha_{12} = 1$. Then $S^{-1} N_\rho S N_\rho^{-1}$ becomes

$$\begin{pmatrix} \rho^{-1} & 0 & 0 \\ (\rho^{-1} - \rho)\,\alpha_{21}\alpha_{23}\alpha_{32} & \rho^{-1}\alpha_{23}\alpha_{31} - \rho\alpha_{21}\alpha_{33} & (\rho - \rho^{-1})\,\alpha_{21}\alpha_{23} \\ \rho\alpha_{33} + \alpha_{21}\alpha_{32}\alpha_{33} - \rho^2\alpha_{31}\alpha_{32}\alpha_{23} & (1 - \rho^2)\,\alpha_{31}\alpha_{33} & \rho^2\alpha_{31}\alpha_{23} - \alpha_{21}\alpha_{33} \end{pmatrix}. \qquad (17)$$

If ρ may take the value -1, (17) multiplies ξ_1 and ξ_2 by -1 and replaces ξ_3 by $\xi_3 - 2\alpha_{33}\xi_1$. Hence it transforms S_1 into an operator $\neq S_a$ of G_{p^3}, whereas the order of H is not a multiple of p^3. Let then $\rho^2 \neq 1$. If $\alpha_{21}\alpha_{23} \neq 0$, (17) falls under case (iii). If $\alpha_{23} = 0$, so that $|S| = -\alpha_{21}\alpha_{33} = 1$, (17) is commutative with J_1 if, and only if, (§3) $\rho^3 = 1$, $\alpha_{31} = 0$; when these hold, $w = 3$, so that $\alpha_{31}' = (\rho - 1)\,\alpha_{32}$ in (17) is zero. In view of S^2 we may set $\alpha_{21} = \epsilon$, $\epsilon^3 = 1$. Then

$$S = \begin{pmatrix} 0 & 1 & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & -\epsilon^2 \end{pmatrix}, \quad S_{-\epsilon^2} S^{-1} S_{\epsilon^2} S N_{\epsilon^2} S_{-1} S = \begin{pmatrix} -\epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ -\epsilon & 2\epsilon^3 & -1 \end{pmatrix},$$

while the latter transforms J_1 into a different subgroup of G_{p^3} (§3). The remaining case $\alpha_{21} = 0$ may be excluded in a similar way.

(iii) $\alpha_{13} = \alpha_{12} = 0$, $\alpha_{23} \neq 0$. Transforming H by N, we may set $\alpha_{23} = 1$. Multiplying right and left by the S_a, we may set also $\alpha_{22} = \alpha_{33} = 0$, whence

$-\alpha_{11}\alpha_{32}=1$. Then $S^{-1}N_\rho S N_\rho^{-1}=L$ is

$$\begin{pmatrix}1 & 0 & 0\\ \beta & \rho^{-1} & 0\\ \gamma & 0 & \rho\end{pmatrix},\quad \begin{array}{l}\beta=(\rho^{-1}-\rho)\,\alpha_{21}\alpha_{32},\\ \gamma=(\rho-\rho^2)\alpha_{31}\alpha_{32}.\end{array}$$

If either $\beta \neq 0$ or $\rho^3 \neq 1$, L leads to a G_{p^3} (§3). Let, then, $\beta=0$, $\rho^3=1$, $\rho\neq 1$ whence $\alpha_{21}=0$. Then $N_\rho^{-1}LN_\rho L^{-1}=B_{3,1,t}$, $t=\gamma(1-\rho^{-1})$, leads to a G_{p^3} unless $\gamma=0$. Hence we may set also $\alpha_{31}=0$. In view of S^3, we may set $\alpha_{32}=\varepsilon$, $\varepsilon^3=1$, whence $\alpha_{11}=-\varepsilon^2$. Then S and $S_{-1}S^{-1}S_\varepsilon SS_{-1}$ are, respectively,

$$\begin{pmatrix}-\varepsilon^2 & 0 & 0\\ 0 & 0 & 1\\ 0 & \varepsilon & 0\end{pmatrix},\quad \begin{pmatrix}1 & 0 & 0\\ -2 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}.$$

The latter, taken as S, leads to an L with $\beta\neq 0$.

10. It remains to consider the case in which J_1 is commutative only with its operators and the $M_{\epsilon,\epsilon,\epsilon}$, with $\epsilon^e=1$, $e=1$ or 3. Denote the order of H by pem. We pass to the quotient-group Q of G by $(M_{\epsilon,\epsilon,\epsilon})$. From H, we obtain H_1 of order pm. Hence H_1 contains exactly m operators of order prime to p. By the canonical form theory, Q contains cyclic subgroups of order $\frac{1}{d}\,(p^2+p+1)$ each commutative with just $\frac{3}{d}\,(p^2+p+1)$ operators. Let μ be the order of the largest subgroup C of one of these cyclic groups which lies in H_1. Then C is one of $pm\div\mu\lambda$ conjugates within H_1, where $\lambda=1$ or 3, and no two of them have a common operator $\neq I$. Hence they contain at least $pm(\mu-1)\div 3\mu$ operators $\neq I$. But if $p>3$, this number exceeds m if $\mu>1$, and hence $\mu \mathrel{\overline{>}} 3$. Hence, for $p>3$, there occur no operators $\neq I$ whose periods divide $\frac{1}{d}\,(p^2+p+1)$. The same is true for $p=3$, since then $d=1$, $\mu=13$, $\lambda=3$, so that there are exactly $m/13$ C_{13} in H_1. But $m=2^i.13$, $i \mathrel{\overline{<}} 4$, contrary to Sylow's theorem. Next, H_1 contains no operators of period τ, a divisor of $\frac{1}{d}(p^2-1)$ but not of $\frac{1}{d}\,(p-1)$. Indeed, the cyclic C_τ would be one of $pm\div\tau x$ conjugates, where $x=1$ or 2. Let C_τ have t operators of periods dividing $\frac{1}{d}\,(p-1)$, $\tau \mathrel{\overline{>}} 2t$. Then there are at least $\tau-t$ operators in any C_τ

occurring in none of its conjugates. But $(\tau - t)pm \div \tau x \gtreqless \frac{1}{2} pm/x > m$ if $p > 3$. The case $p = 3$ is immediately treated since the group of order $3 \cdot 2^i$, $1 \lesseqgtr i \lesseqgtr 4$ has 2^i conjugate C_3 and a self-conjugate G_{2^i}, whence $i = 2$ or 4. If $i = 2$ and the G_4 is cyclic, there occurs a self-conjugate operator O_2 and hence an O_6. For $i = 4$, G_{16} must contain an O_8, since 16 is the highest power of 2 dividing the order of $G \equiv Q$, which contains operators of period $\frac{1}{d}(p^2 - 1) = 8$. But for $\tau = 8$, $t = 2$, the general argument gives $6 \cdot 3 \cdot m \div 8x$, or more than m, distinct operators of periods 4 and 8.

We have shown that H_1 contains no operator of period a divisor > 1 of $q = \frac{1}{d}(p^2 + p + 1)$, or a divisor of $r = \frac{1}{d}(p^2 - 1)$ but not of $s = \frac{1}{d}(p - 1)$. The order of Q is $q(p+1)(p-1)^2 p^3$. Now q is relatively prime to $p + 1$ Also $(p-1)^2 - dq = -3p$, while q is not divisible by 3; hence q and $(p-1)^2$ are relatively prime. Hence the order of H_1 divides $(p+1)(p-1)^2 p$. Any factor other than 2 or 4 of $p + 1$ is prime to $(p-1)^2$ and hence to s. Hence the order of H_1 divides $w = 2^k (p-1)^2 p$, $k = 2$ if $p = 4l + 1$, $k = 4$ if $p = 4l + 3$. The order must divide $w/2$; otherwise H_1 would contain a group of order the highest power of 2 dividing the order of Q, and hence an operator of period a power of 2 dividing r but not s. Hence the order H_1 divides $v = x(p-1)^2 p$, $x = 1$ or 2 according as $p = 4l \pm 1$. But the only divisors $\equiv 1 \pmod{p}$ of $2(p-1)^2$ are 1, $(p-1)^2$, and if $p = 7$ also 8. But C_p is to be commutative with no further operators of H_1. If $p = 7$ and H_1 is of order 56, there would occur an abelian subgroup G_8 of type (1, 1, 1), whereas a simple discussion shows that no such G_8 lies in $LF(3, 7)$. Hence H_1 is of order $p(p-1)^2$. Now $(p-1)^2$ has no factor $\equiv 1 \pmod{p}$ other than itself and unity. Moreover, C_p is not self-conjugate. Hence* $(p-1)^2$ does not have two distinct prime factors. Hence $p - 1 = 2^a$. Then $d = 1$, $\frac{1}{2}(p+1)$ is odd. Hence H_1 contains a $G_{2^{2a}}$ self-conjugate in a subgroup $G_{2^{2a+1}}$ of order the highest power of 2 in $LF(3, p)$. The latter has operators of period $p^2 - 1$, and hence operators of period 2^{a+1}. Hence $G_{2^{2a}}$ has operators of period 2^a. But no operator of period p transforms into itself a cyclic C_{2^i} $(i \lesseqgtr a)$, since there is no ternary group of order $p2^i$ containing J_1. Hence there are at least p distinct conjugates C_{2^i} in H_1

* Frobenius, Berliner Sitzungsberichte, 1902, p. 459.

and hence at least $p2^{i-1}$ distinct O_i. But

$$p\sum_{i=1}^{a} 2^{i-1} = p(2^a - 1) = 2^{2a} - 1.$$

Hence H_1 contains exactly p conjugate C_{2^i} for $i = 1, \ldots, a$. If two of the C_{2^a} had a common subgroup C_{2^b}, $b > 0$, all the p C_{2^a} would contain C_{2^b}, which would then be self-conjugate in H_1, contrary to the above. Hence all the operators of $G_{2^{2a}}$ lie in p conjugate G_{2^a}. Suppose that exactly 2^c operators transform each of the p cyclic C_{2^i} into itself. Then there would be a self-conjugate G_{2^c} in H_1. Hence, by above, $c = 2a$, so that $G_{2^{2a}}$ is abelian of type (a, a). Hence there are only 3 operators of period 2, whence $p = 3$. For $p = 3$, $H_1 = H$ is simply isomorphic with the alternating group on 4 letters. Its four-group may be taken, by applying a suitable ternary transformation, to be generated by $M_{-1,-1,1}$ and B_+, where

$$B_{\pm} = \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_{11} & \pm\alpha_{11} & \alpha_{13} \\ \mp\alpha_{11} & -\alpha_{11} & \pm\alpha_{13} \\ \alpha_{31} & \mp\alpha_{31} & 0 \end{pmatrix}, \quad 4\alpha_{11}\alpha_{13}\alpha_{31} \equiv 1.$$

We find that any ternary operator of determinant 1 which transforms $M_{-1,-1,1}$ into $B_{\pm}$, and the latter into $B_{\mp}$, must be of the form C. Every such C is of period 3 and leaves absolutely invariant

$$\xi_1^2 + \xi_2^2 - \xi_3^2 \equiv \xi_2^2 - 2(\xi_1 + \xi_3)(\xi_1 - \xi_3), \qquad (\text{mod } 3).$$

The resulting G_{12} generated by $M_{-1,-1,1}$, B_+, and any C, is, therefore, conjugate within G with the group of the operators (16).

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Deduction of the Power Series Representing a Function from Special Values of the Latter.

BY G. W. HILL.

I have already treated this matter in another place,* but the exposition there is by illustration only and quite incomplete. The subject needs a more general presentation, which will be the endeavor here.

The treatment of the question is much facilitated or, in many cases, even rendered possible, by the application of two principles. The first is the isolation of groups in the assemblage of linear equations through the attribution of zero values to some of the parameters involved. The second is the disintegration of the equations by comparison when corresponding positive and negative values are given to one or more of the parameters.

Here it is expedient to adopt a peculiar notation. Let F denote the function to be treated and x the general parameter. The formulæ to be written in what follows will be limited to the case where there are four parameters; the modifications to be made when there are more or less will be obvious. We use i for the general integral exponent always not negative, and A for the general coefficient. There is here no necessity for the employment of accents or subscripts to distinguish quantities of the same kind. The parameters will be known as the first, second, third and fourth. In designating any one of these all must be written; thus, the third parameter is $x^0x^0xx^0$. Accordingly, we write the equation

$$F = \Sigma A x^i x^i x^i x^i,$$

* Astronomical Journal, No. 567.

where the i's are not necessarily the same. Let subscripts attached to F denote the special values of the function correspondent to special values of the parameters; and, as we have to distinguish between significant and zero values for the latter, let us suppose that i always denotes a positive integer; consequently, the value $i = 0$ is excluded from the summations Σ. The function F must undergo a sort of differencing in reference to zero values for the parameter; a differencing which is more general than the ordinary, since it often involves more than one variable. This mode of operating is, for the adopted case, depicted in the following system of equations:

$$\Sigma \,.\, Ax^0x^0x^0x^0 = F_{0000} = \overset{0}{F}_{0000},$$

$$\left\{\begin{aligned}
\Sigma \,.\, Ax^ix^0x^0x^0 &= F_{x000} - F_{0000} = \overset{1}{F}_{x000},\\
\Sigma \,.\, Ax^0x^ix^0x^0 &= F_{0x00} - F_{0000} = \overset{1}{F}_{0x00},\\
\Sigma \,.\, Ax^0x^0x^ix^0 &= F_{00x0} - F_{0000} = \overset{1}{F}_{00x0},\\
\Sigma \,.\, Ax^0x^0x^0x^i &= F_{000x} - F_{0000} = \overset{1}{F}_{000x},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\Sigma \,.\, Ax^ix^ix^0x^0 &= F_{xx00} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00} - F_{0000} = \overset{2}{F}_{xx00},\\
\Sigma \,.\, Ax^ix^0x^ix^0 &= F_{x0x0} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{x0x0},\\
\Sigma \,.\, Ax^ix^0x^0x^i &= F_{x00x} - \overset{1}{F}_{x000} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{x00x},\\
\Sigma \,.\, Ax^0x^ix^ix^0 &= F_{0xx0} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{0xx0},\\
\Sigma \,.\, Ax^0x^ix^0x^i &= F_{0x0x} - \overset{1}{F}_{0x00} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{0x0x},\\
\Sigma \,.\, Ax^0x^0x^ix^i &= F_{00xx} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{00xx},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\Sigma \,.\, Ax^ix^ix^ix^0 &= F_{xxx0} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{0xx0} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00}\\
&\qquad - \overset{1}{F}_{00x0} - F_{0000} = \overset{3}{F}_{xxx0},\\
\Sigma .\, Ax^ix^ix^0x^i &= F_{xx0x} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x00x} - \overset{2}{F}_{0x0x} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00}\\
&\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{xx0x},\\
\Sigma \,.\, Ax^ix^0x^ix^i &= F_{x0xx} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{x00x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0}\\
&\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{x0xx},\\
\Sigma \,.\, Ax^0x^ix^ix^i &= F_{0xxx} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0}\\
&\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{0xxx},
\end{aligned}\right.$$

$$
\begin{aligned}
\Sigma \,.\, Ax^i x^i x^i x^i = F_{xxxx} - \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{3}{F}_{x0xx} - \overset{3}{F}_{0xxx} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0} \\
- \overset{2}{F}_{x00x} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000} \\
- \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} \qquad = \overset{4}{F}_{xxxx}.
\end{aligned}
$$

The number of these equations is $16 = 2^4$, and, generally, if there are k parameters, the number is 2^k. It will be readily perceived that $\overset{0}{F}$ is the term of the series independent of the parameters; that the $\overset{1}{F}$ are functions of the single significant parameter appearing in their subscripts, without a term independent of that parameter; that the $\overset{2}{F}$ are functions of the two significant parameters appearing in their subscripts, without any terms independent of one or both parameters; that the $\overset{3}{F}$ are functions of the three significant parameters appearing in their subscripts, without any terms independent of one, two or all of these parameters; and, finally, that $\overset{4}{F}$ is a function of all four parameters, but without any terms independent of one, two, three or all of the parameters. Thus each F of a definite superscript involves no terms included in the F of smaller superscripts. By this device we have broken the system of linear equations for the determination of the coefficients into 16 groups, each of which can be treated independently of the others.

It is not necessary that the computations should be made by the equations just written. The last involves no less than 16 terms, and labor will be saved by eliminating some of the F. The 5 equations at the beginning remaining unmodified, it will be perceived the following system is equivalent to the former:

$$
\left\{
\begin{aligned}
\overset{2}{F}_{xx00} &= F_{xx00} - \overset{1}{F}_{x000} - F_{0x00}, \\
\overset{2}{F}_{x0x0} &= F_{x0x0} - \overset{1}{F}_{00x0} - F_{x000}, \\
\overset{2}{F}_{x00x} &= F_{x00x} - \overset{1}{F}_{000x} - F_{x000}, \\
\overset{2}{F}_{0xx0} &= F_{0xx0} - \overset{1}{F}_{00x0} - F_{0x00}, \\
\overset{2}{F}_{0x0x} &= F_{0x0x} - \overset{1}{F}_{000x} - F_{0x00}, \\
\overset{2}{F}_{00xx} &= F_{00xx} - \overset{1}{F}_{00x0} - F_{000x},
\end{aligned}
\right.
$$

$$\begin{cases} \overset{3}{F}_{xxx0} = F_{xxx0} - \overset{2}{F}_{xx00} - F_{x0x0} - F_{0xx0} + F_{00x0}, \\ \overset{3}{F}_{xx0x} = F_{xx0x} - \overset{2}{F}_{xx00} - F_{0x0x} - F_{x00x} + F_{000x}, \\ \overset{3}{F}_{x0xx} = F_{x0xx} - \overset{2}{F}_{00xx} - F_{x00x} - F_{x0x0} + F_{x000}, \\ \overset{3}{F}_{0xxx} = F_{0xxx} - \overset{2}{F}_{00xx} - F_{0x0x} - F_{0xx0} + F_{0x00}, \end{cases}$$

$$\overset{4}{F}_{xxxx} = F_{xxxx} - \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{2}{F}_{xx00} - F_{x0xx} - F_{0xxx} + F_{00xx}.$$

These formulæ are not the unique ones of their type, but the $\overset{2}{F}$ admit two different forms, the $\overset{3}{F}$ three and $\overset{4}{F}$ six. All are obtained by making certain transpositions between the x and 0 of the subscripts. They need not be given here, as their employment has no advantage over those just written.

Each of the $\overset{i}{F}$ is evidently divisible by the product of the significant parameters in its subscript. The functions thus obtained may be considered as one step nearer the result of elimination. We may use G to denote them. Thus:

$$G_{0000} = \frac{1}{x^0x^0x^0x^0} F_{0000}, \qquad G_{x000} = \frac{1}{xx^0x^0x^0} \overset{1}{F}_{x000}, \qquad G_{0x00} = \frac{1}{x^0xx^0x^0} \overset{1}{F}_{0x00}, \text{ etc.,}$$

$$G_{xx00} = \frac{1}{xxx^0x^0} \overset{2}{F}_{xx00}, \text{ etc.,} \quad G_{xxx0} = \frac{1}{xxxx^0} \overset{3}{F}_{xxx0}, \text{ etc.,} \quad G_{xxxx} = \frac{1}{xxxx} \overset{4}{F}_{xxxx}.$$

We come now to the application of the second principle. In the first place consider F when involving only a single significant parameter as F_{x000}, and let F_{+000} and F_{-000} denote the values of F_{x000} for corresponding positive and negative values of x; then it is plain we shall have

$$\Sigma A\,(x^2)^i\, x^0x^0x^0 = \tfrac{1}{2}\,[F_{+000} + F_{-000}],$$
$$xx^0x^0x^0\, \Sigma A\,(x^2)^i\, x^0x^0x^0 = \tfrac{1}{2}\,[F_{+000} - F_{-000}],$$

where the A of the first equation are distinct from the A of the second, and where it is now necessary to allow i to assume the value 0.

Next, supposing F involves two significant parameters as F_{xx00}, then we shall have the four equations

$$\Sigma A\,(x^2)^i\, xx^0x^0 = \tfrac{1}{2}\,[F_{+x00} + F_{-x00}],$$
$$xx^0x^0x^0 \Sigma A\,(x^2)^i\, xx^0x^0 = \tfrac{1}{2}\,[F_{+x00} - F_{-x00}],$$
$$\Sigma Ax\,(x^2)^i\, x^0x^0 = \tfrac{1}{2}\,[F_{x+00} + F_{x-00}],$$
$$x^0xx^0x^0 \Sigma Ax\,(x^2)^i\, x^0x^0 = \tfrac{1}{2}\,[F_{x+00} = F_{x-00}].$$

By taking half the sum and half the difference of certain of these, we obtain the four equations

$$\begin{aligned}
\Sigma A x^{2i} x^{2i} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} + F_{+-00} + F_{-+00} + F_{--00}\right], \\
\Sigma A x^{2i+1} x^{2i} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} + F_{+-00} - F_{-+00} - F_{--00}\right], \\
\Sigma A x^{2i} x^{2i+1} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} - F_{+-00} + F_{-+00} - F_{--00}\right], \\
\Sigma A x^{2i+1} x^{2i+1} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} - F_{+-00} - F_{-+00} + F_{--00}\right],
\end{aligned}$$

where the coefficients A are distinct for each. The second, third and fourth are divisible severally by $xx^0x^0x^0$, $x^0xx^0x^0$ and xxx^0x^0. By making these divisions we shall be a step nearer the result of elimination. The rule of the signs connecting the form F in the group of four equations may seem a little obscure, but a consideration of the successive operations of taking half the sum and difference shows that the sign of each F is given by raising the signs in the subscripts to the same powers as the corresponding parameters have in the left members of the equations. As heré, the even integer $2i$ may be dropped out of the exponents, we perceive that the signs in question are given by the expressions

$$\begin{array}{llll}
(+)^0(+)^0, & (+)^0(-)^0, & (-)^0(+)^0, & (-)^0(-)^0, \\
(+)^1(+)^0, & (+)^1(-)^0, & (-)^1(+)^0, & (-)^1(-)^0, \\
(+)^0(+)^1, & (+)^0(-)^1, & (-)^0(+)^1, & (-)^0(-)^1, \\
(+)^1(+)^1, & (+)^1(-)^1, & (-)^1(+)^1, & (-)^1(-)^1.
\end{array}$$

In case F involves three significant parameters, as F_{xxx0}, we have entirely analogous equations which, for brevity, we write as follows:

$$\begin{aligned}
\Sigma A x^{2i} x^{2i} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^0(\pm)^0 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i+1} x^{2i} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^0(\pm)^0 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i} x^{2i+1} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^1(\pm)^0 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i} x^{2i} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^0(\pm)^1 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i+1} x^{2i+1} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^1(\pm)^0 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i+1} x^{2i} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^0(\pm)^1 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i} x^{2i+1} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^1(\pm)^1 F_{\pm\pm\pm 0}, \\
\Sigma A x^{2i+1} x^{2i+1} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^1(\pm)^1 F_{\pm\pm\pm 0}.
\end{aligned}$$

The connection of the ambiguous signs in these equations will be readily understood.

In case F has all its parameters significant, there are 16 equations analogous to the preceding; we need write but one as a type of all:

$$\Sigma A x^{2i} x^{2i} x^{2i} x^{2i} = \tfrac{1}{2^4} S (\pm)^0 (\pm)^0 (\pm)^0 (\pm)^0 F_{\pm\pm\pm\pm}.$$

Thus, by the application of the two principles of zero values and of pairs of values of opposite signs, we succeed in breaking the system of linear equations to be solved into several subordinate systems entirely independent of each other. When the number of parameters is 2, it is evident that the number of these subordinate systems is

$$1 . 1 + 2 . 2 + 4 . 1 = 9 = 3^2;$$

where there are 3 parameters this number is

$$1 . 1 + 2 . 3 + 4 . 3 + 8 . 1 = 27 = 3^3;$$

and when the number of parameters is 4 (the case we have been treating) the number is

$$1 . 1 + 2 . 4 + 4 . 6 + 8 . 4 + 16 . 1 = 81 = 3^4;$$

hence, in the general case, where there are k parameters, the number of independent subordinate systems is 3^k.

After having shown the applicability of zero and parity values of the parameters for breaking the system of linear equations into detached portions, it remains to show what principles should guide us in selecting the values of the parameters for which the special values of the function are to be computed. As in the former memoir we suppose that the values of each parameter are taken from an arithmetical progression of which one term is zero. Let d denote the common difference in this progression which, although it may be different for each parameter, we designate by the same letter, just as before we employed x and i. Then, in the first instance, the power series will be derived in the form

$$F = \Sigma A \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i.$$

As it is necessary to cut off the power series at some limit, it is desirable to choose the d in such a way that the neglected terms should vitiate as little as possible the derived values of the A. The smaller are the d the smaller is this vitiation; but practical considerations set a limit to this diminution. Suppose we are going to quantities of the 10^{th} order of smallness in the x, and decide to halve the d; then, as $2^{10} = 1024$, it will be necessary to add 3 more decimals in our computations; and if the d are diminished to a tenth, 10 decimals must be added, which procedure could not generally be entertained. Thus nice judgment is required in deciding on the magnitude of the d. As good a rule for the choice as can be given is to divide the range over which the parameter is supposed to play by the number of significant exponents it is to receive in the power series. Then the selection of the values of the parameters should be such that, in a graphical exhibition, they would be arranged as nearly as possible in a symmetrical manner about the origin; and, in a space of k dimensions if k is the number of parameters, they should be contained within the ellipsoid whose axes are the several ranges.

As the computation of special values of the function constitutes much the larger part of the labor incident to the method, it is desirable to insist on the limitation that no more special values are to be computed than terms are to be retained. But some restrictions must be put on the employment of the two principles given for the purpose of disintegrating the system of linear equations, and on the selection of values of the parameters for which the special values of the function are to be computed.

If F is computed for $x = id$, $x = id$, $x = id$, $x = id$, we shall call $iiii$ the argument of the value of F. Here i is integral, but may be zero or negative. Then, in each group of linear equations obtained by the application of the first principle, it is plain that the zeros of the arguments used must fall in the same place as the zero exponents of the parameters; thus, when we are treating the group whose type is $[i00i]$, the arguments of the special values used must be of the type $i00i$, where, however, i can be negative. The first principle can always be used, but it is desirable to limit the selection of arguments in the following manner:—Dividing the terms into Division I, where all the exponents are zeros, Division II, when all but one are zeros, and III where all but two are zeros, and so on; if we have used an argument such as $iii0$ in Division IV, it is necessary to use the arguments $ii00$, $i0i0$, $0ii0$ in the preceding or here Division III,

understanding that that the i in the second case are identical with those standing in the same place in the first. Hence the proper method of selecting the arguments to be used seems to be to commence at Division I, for which, in the case we exhibit, the argument is 0000, and get the arguments for Division II by substituting for one of the zeros an integer positive or negative. Then the arguments for Division III are got from these by substituting for one of the remaining zeros positive or negative integers, and so on to the end. These integers should constitute in each case an arithmetical progression having zero near the middle of it.

With regard to the application of the second principle, that of parity values it often cannot be employed without introducing non-independent equations. The remedy for this state of things is to cut down the operations to a half stage or even to a quarter stage, and, in some cases, not to employ it at all.

These matters cannot be well set forth without the help of an example. We adopt that of the preceding memoir. It is characterized by saying that it involves four parameters, two of which are regarded as of the first, and two of the second order of smallness; and all terms above the eighth order are to be neglected. This demands the presence of 175 terms in the power series. How they are disintegrated into 81 subgroups by the application of our two principles is shown in the following table. As each term is sufficiently characterized by the exponents of the four parameters, nothing else is set down, and the terms of each subgroup are connected by the sign +. In addition, the exponents of the first term are set down as they are, but the following terms of the line are divided by the first term, as the quotients are more useful than the terms themselves.

Division.	Group.	Sub-group.	Divisor.	Quotients.
I	1	1	0000	
II	2	2	2000	+2000+4000+6000
		3	1000	" " "
	3	4	0200	+0200+0400+0600
		5	0100	" " "
	4	6	0020	+0020
		7	0010	"
	5	8	0002	+0002
		9	0001	"
III	6	10	2200	+2000+0200+4000+2200+0400
		11	1200	" " " " "
		12	2100	" " " " "
		13	1100	+ " " " " " +6000+4200+2400+0600
	7	14	2020	"
		15	1020	"
		16	2010	" +0020 "
		17	1010	" " "
	8	18	2002	"
		19	1002	"
		20	2001	" +0002 "
		21	1001	" " "
	9	22	0220	+0200
		23	0120	"
		24	0210	" +0020+0400
		25	0110	" " "
	10	26	0202	"
		27	0102	"
		28	0201	" +0002 "
		29	0101	" " "
	11	30	0022	
		31	0012	
		32	0021	
		33	0011	+0020 '

Division.	Group.	Sub-group.	Divisor.	Quotient.
		34	2220	
		35	1220	
		36	2120	
	12	37	2210	+2000+0200
		38	1120	" "
		39	1210	" "
		40	2110	" "
		41	1110	" " +0020+4000+2200+0400
		42	2202	
		43	1202	
		44	2102	
	13	45	2201	" "
		46	1102	" "
		47	1201	" "
IV		48	2101	" "
		49	1101	" " +0002 " " "
		52	2012	
		53	2021	
	14	54	1012	
		55	1021	
		56	2011	"
		57	1011	"
		60	0212	
		61	0221	
	15	62	0112	
		63	0121	
		64	0211	+0200
		65	0111	"
		76	2211	
		77	2111	
V	16	78	1211	
		79	1121	
		80	1112	
		81	1111	+2000 "

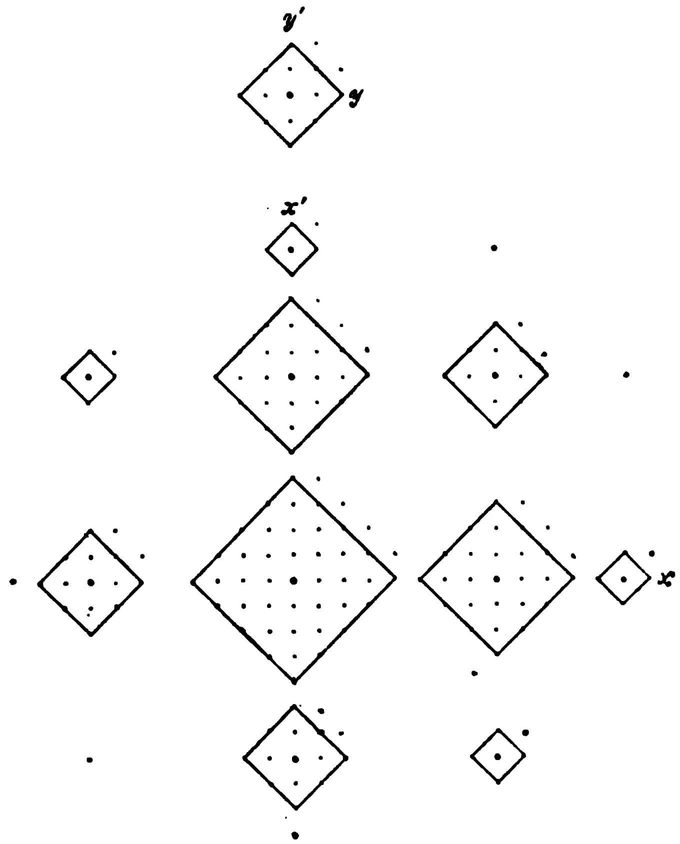
The 16 groups in the table are the result of the application of the first principle; the 81 sub-groups result from the further application of the second principle. It will be noticed that 14 out of the 81 sub-groups do not appear in the table; this is because their terms are all above the 8^{th} order. The success of the application of the two principles is well shown by the table. Out of 81 sub-groups there is only one (the 13^{th}) which consists of as many as 10 equations and 10 unknowns; two groups have 7, and three have 6; and 23 groups consist of a single equation giving the value of one coefficient each.

The following table shows a selection of arguments which may be employed in our illustrative example. The ambiguous signs must be taken in every possible combination; thus three in one argument denote eight different arguments.

Division.	Group.	Arguments.									
I	1	0 0 0 0									
II	2	±1 0 0 0	±2 0 0 0	±3 0 0 0	±4 0 0 0						
	3	0 ±1 0 0	0 ±2 0 0	0 ±3 0 0	0 ±4 0 0						
	4	0 0 ±1 0	0 0 ±2 0								
	5	0 0 0 ±1	0 0 0 ±2								
III	6	±1 ±1 0 0	±2 ±1 0 0	±1 ±2 0 0	±2 ±2 0 0	±3 ±1 0 0	±1 ±3 0 0	1 4 0 0	2 3 0 0	3 2 0 0	4 1 0 0
	7	±1 0 ±1 0	±2 0 ±1 0	±1 0 2 0	±3 0 1 0						
	8	±1 0 0 ±1	±2 0 0 ±1	±1 0 0 2	±3 0 0 1						
	9	0 ±1 ±1 0	0 ±2 ±1 0	0 ±1 2 0	0 ±3 1 0						
	10	0 ±1 0 ±1	0 ±2 0 ±1	0 ±1 0 2	0 ±3 0 1						
	11	0 0 ±1 ±1	0 0 2 1	0 0 1 2							
IV	12	±1 ±1 ±1 0	±2 ±1 1 0	±1 ±2 1 0	2 1 −1 0	1 2 −1 0	2 2 1 0	3 1 1 0	1 3 1 0	1 1 2 0	
	13	±1 ±1 0 ±1	±2 ±1 0 1	±1 ±2 0 1	2 1 0 −1	1 2 0 −1	2 2 0 1	3 1 0 1	1 3 0 1	1 1 0 2	
	14	±1 0 ±1 1	±1 0 1 −1	±2 0 1 1							
	15	0 ±1 ±1 1	0 ±1 1 −1	0 ±2 1 1							
V	16	±1 ±1 1 1	1 1 −1 1	1 1 1 −1	2 1 1 1	1 2 1 1					

It will be seen from this table that the second principle has not in every case been pushed to its limit. Thus in Div. IV, Group 12, if we employ the 8 arguments $\pm 1\ \pm 1\ \pm 1\ \ 0$ we get as many independent relations between the sought

coefficients; but, if we annex the 8 augments $\pm 2 \pm 1 \pm 1\ 0$, we do not get 8 additional relations but only 5. This is explained by the fact (consult the arrangement of terms in Group 12 in the first table) that the first 8 give the values of 3 coefficients, and the second 8 also give them.



We will catalogue all the deviations from a complete parity treatment in the foregoing table. In Group 6, the parity treatment, here involving two steps, has been applied only in six cases, while four arguments are without it; to have applied it to the latter would have introduced superfluous relations. In Groups 7-10 we have two instances of parity treatment to two steps, and two to one step. In Group 11 one instance of this treatment to two steps and two arguments without it. In Groups 12 and 13 one instance to three steps, two to two and six arguments without it. In Groups 14 and 15 one instance to two steps

and two to one. In fine, in Group 16 one instance to two steps and four arguments without it.*

But it is much easier to comprehend the principles which should be followed in the choice of the arguments through a graphical exhibition. The 175 arguments in our example, since they are to four elements, can be represented in a space of four dimensions. By drawing in this space $3.5 = 15$ planes properly chosen, the points representing the arguments will all lie in these planes. We adopt here for the coordinates the notation of the first memoir, viz., $xx'yy'$. In the adjacent diagram the upper oblique square with its two adjacent points constitutes a table of contents or index to the graphs of the 15 planes shown below; it bears on the coordinates y and y' or the third and fourth constituents of the argument. These graphs are placed relatively to each other as the points of the index which belong to them. By this device we are enabled to represent on a plane, sufficiently for our purposes, a space of four dimensions. Moreover, the graphs are placed so as not to interfere with each other, the coordinates x and x' being measured from the central point of the oblique squares. The introduction of the latter into the diagram has no other object than to enable the eye to grasp quickly the law of distribution of the points.

It will be perceived that 5 of the graphs reduce to a single point; they may be called oblique squares to side 0. Next 4 graphs consist of oblique squares to side 1 and they all have one point exterior to the square. Again there are 3 graphs to side 2 with 2 exterior points. Next 2 graphs to side 3 with 3 exterior points; and finally, a single graph to side 4 with 4 exterior points. With regard to these exterior points, it must be explained that the positions they have in the diagram are not unique. Let us suppose that the positions lying nearest the perimeter of an oblique square and exterior to it are called the adjacent points; they are in number four times the number expressing the side of the square, and they can be joined by straight lines so as to form rectangles. Then the exterior points must be distributed in such a way that each rectangle shall receive one and but one point at some one of its angles. It is not necessary that a similar arrangement should be adopted for all or for some of the graphs; it may be varied at will. In the diagram the exterior points are, in all cases, placed to the

* The choice of arguments made here for the 175 special values of F is not quite the same as that in the first memoir. If, in the 8 arguments numbered 120-123, 142-145, 1 is substituted for 3 in the first constituent, we have the present selection.

upper and right side of the square. As to the arrangement of the squares in reference to the magnitude of their sides, it will be perceived that on the one hand the limit is a square of the side 0, and on the other a square of side 1; and, as we pass inwards towards the centre, at every step the side augments by 2; but when we arrive at the middle column, it is only a half-step on the right hand, while it is a whole step on the left. This is for an even number of parameters; for an odd number, the half steps do not exist.

The number of points in each graph is shown by the following scheme:

$$\left.\begin{array}{l} 1.1 \\ 1.1+2.3 \\ 1.1+2.3+3.5 \\ 1.1+2.3+3.5+4.7 \\ 1.1+2.3+3.5+4.7+5.9 \end{array}\right\} = 175.$$

The regularity apparent in the diagram is due to the tabulation of the points under the headings of two of the parameters. However, after the diagram is formed, it will not be difficult to distribute the arguments under the headings of the groups. It will be noticed that the exterior points are each a half-unit distant from the perimeters of the squares. As we have placed them they may be included in a rectangle having one more column in one direction than in the other.

When we have the arguments of the special values which determine the coefficients of a sub-group, it is easy to write, with the assistance of the first table, the determinant belonging to the solution. Thus, in Sub-group 13 of our example the determinant is

$$\left|\begin{array}{cccccccccc}
1^0.1^0 & 1^2.1^0 & 1^0.1^2 & 1^4.1^0 & 1^2.1^2 & 1^0.1^4 & 1^6.1^0 & 1^4.1^2 & 1^2.1^4 & 1^0.1^6 \\
2^0.1^0 & 2^2.1^0 & 2^0.1^2 & 2^4.1^0 & 2^2.1^2 & 2^0.1^4 & 2^6.1^0 & 2^4.1^2 & 2^2.1^4 & 2^0.1^6 \\
1^0.2^0 & 1^2.2^0 & 1^0.2^2 & 1^4.2^0 & 1^2.2^2 & 1^0.2^4 & 1^6.2^0 & 1^4.2^2 & 1^2.2^4 & 1^0.2^6 \\
2^0.2^0 & 2^2.2^0 & 2^0.2^2 & 2^4.2^0 & 2^2.2^2 & 2^0.2^4 & 2^6.2^0 & 2^4.2^2 & 2^2.2^4 & 2^0.2^6 \\
3^0.1^0 & 3^2.1^0 & 3^0.1^2 & 3^4.1^0 & 3^2.1^2 & 3^0.1^4 & 3^6.1^0 & 3^4.1^2 & 3^2.1^4 & 3^0.1^6 \\
1^0.3^0 & 1^2.3^0 & 1^0.3^2 & 1^4.3^0 & 1^2.3^2 & 1^0.3^4 & 1^6.3^0 & 1^4.3^2 & 1^2.3^4 & 1^0.3^6 \\
4^0.1^0 & 4^2.1^0 & 4^0.1^2 & 4^4.1^0 & 4^2.1^2 & 4^0.1^4 & 4^6.1^0 & 4^4.1^2 & 4^2.1^4 & 4^0.1^6 \\
3^0.2^0 & 3^2.2^0 & 3^0.2^2 & 3^4.2^0 & 3^2.2^2 & 3^0.2^4 & 3^6.2^0 & 3^4.2^2 & 3^2.2^4 & 3^0.2^6 \\
2^0.3^0 & 2^2.3^0 & 2^0.3^2 & 2^4.3^0 & 2^2.3^2 & 2^0.3^4 & 2^6.3^0 & 2^4.3^2 & 2^2.3^4 & 2^0.3^6 \\
1^0.4^0 & 1^2.4^0 & 1^0.4^2 & 1^4.4^0 & 1^2.4^2 & 1^0.4^4 & 1^6.4^0 & 1^4.4^2 & 1^2.4^4 & 1^0.4^6
\end{array}\right|$$

There is no need of proving that these determinants are non-vanishing, as they are all met with in the problem of drawing a parabolic curve through a definite number of distinct points in a space of two or more dimensions.

On the Definition of Reducible Hypercomplex Number Systems.[*]

BY SAUL EPSTEEN AND HEMAN BURR LEONARD.

§1.—*Preliminary Remarks.*

A hypercomplex number system is said to be reducible[†] when, by a proper choice of the units

$E \equiv E_j\ E_k \equiv e_1 \ldots . e_m e_{m+1} \ldots . e_n$, with the relations $e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}$,

the following conditions are fulfilled:

C_1), E_j forms a system by itself,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3}, \qquad (\gamma_{j_1 j_2 k} = 0);$$

C_2), E_k forms a system by itself,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3}, \qquad (\gamma_{k_1 k_2 j} = 0);$$

C_{jk}), $\qquad e_j e_k = 0, \quad j = 1, \ldots . , m, \qquad (\gamma_{jki} = 0);$

C_{kj}), $\qquad e_k e_j = 0, \quad k = m+1, \ldots . , n, \quad (\gamma_{kji} = 0).$

The system E is supposed to be associative. We add moreover conditions concerning division.

A), associativity;

C_r), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero;

C_l), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero.

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[*] Read before the Chicago Section of the American Mathematical Society, April 2, 1904.

[†] Benj. Peirce, American Journal of Mathematics, vol. 4 (1881), p. 100; Scheffers, Mathematische Annalen, vol. 39 (1891), p. 317.

In the previous paper of the same title* it was shown that the conditions C_1 and C_2 are consequences of

(1) A, C_{jk}, C_{kj}, C_r, or of
(2) A, C_{jk}, C_{kj}, C_l.

In (1) and (2) *the assumptions* A, C_r, C_l *are characterized by the fact that they refer to the system* E *as a whole.*

Adopting suggestions of Professor E. H. Moore the associativity condition A is separated into eight parts and the conditions C_{jk} and C_{kj} are each separated into two parts.

The general number $X_p = \sum_{i_1=1}^{n} x_{pi_1} e_{i_1}$ of the system E is the sum of two components

$$X_p = J_p + K_p = \sum_{j_1=1}^{m} x_{pj_1} e_{j_1} + \sum_{k_1=m+1}^{n} x_{pk_1} e_{k_1}.$$

The condition A: $(X_1 X_2) X_3 = X_1 (X_2 X_3)$ is equivalent to the following eight:

$$\begin{array}{ll}
A_1), & (J_1 J_2) J_3 = J_1 (J_2 J_3); \\
A_2), & (K_1 K_2) K_3 = K_1 (K_2 K_3); \\
A_3), & (K_1 J_1) J_2 = K_1 (J_1 J_2); \\
A_4), & (J_1 K_1) K_2 = J_1 (K_1 K_2); \\
A_5), & (J_1 K_1) J_2 = J_1 (K_1 J_2); \\
A_6), & (K_1 J_1) K_2 = K_1 (J_1 K_2); \\
A_7), & (J_1 J_2) K_1 = J_1 (J_2 K_1); \\
A_8), & (K_1 K_2) J_1 = K_1 (K_2 J_1).
\end{array}$$

In general $J_1 K_1 = J_2 + K_2$. The condition C_{jk} says that $\left\{\begin{array}{l} J_2 = 0 \\ K_2 = 0 \end{array}\right\}$ simultaneously. Thus it is seen that C_{jk} is equivalent to the two conditions:

$$\begin{array}{ll}
C^j_{jk}), & J_1 K_1 = 0 + K_2 = K_2; \\
C^k_{jk}), & J_1 K_1 = J_2 + 0 = J_2.
\end{array}$$

In general $K_3 J_3 = J_4 + K_4$. Similarly then, C_{kj} says that $\left\{\begin{array}{l} J_4 = 0 \\ K_4 = 0 \end{array}\right\}$

* Epsteen, Transactions of the American Mathematical Society, vol. 5 (1904), p. 105.

simultaneously; and thus C_{kj} is equivalent to the two conditions:

$$C^j_{kj}),\qquad K_3J_3 = 0 + K_4 = K_4\,;$$
$$C^k_{kj}),\qquad K_3J_3 = J_4 + 0 = J_4\,.$$

The condition for right-hand division may be expressed thus:

C_r), not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1X = 0, \text{ only if } X_1 = 0\,.$$

The condition for left-hand division may be expressed thus:

C_l), not every X is a left-hand divisor of zero; hence, an X exists such that

$$XX_1 = 0\,, \text{ only if } X_1 = 0.$$

When the condition C_1 is not true, then there are at least two numbers of the set of units E_j, namely J_1 and J_2, such that their product

$$J_1J_2 = J_3 + K_3 \qquad\qquad (K_3 \neq 0).$$

And likewise if C_2 is not true, there are at least two numbers K_1 and K_2 of the set of units E_k, such that their product

$$K_1K_2 = J_3 + K_3 \qquad\qquad (J_3 \neq 0).$$

The condition C^j_r which is employed in this paper means that

C^j_r), not every J is a right-hand divisor of zero in the set E_j; hence, a J exists such that

$$J_1J = 0, \text{ only if } J_1 = 0\,.$$

Similarly C^j_l means that

C^j_l), not every J is a left-hand divisor of zero in the set E_j; hence, a J exists such that

$$JJ_1 = 0, \text{ only if } J_1 = 0\,.$$

From the preceding the meaning of C^k_r and C^k_l is evident,

We have, therefore, to consider in this paper the following twenty conditions:

A_1), $(J_1 J_2) J_3 = J_1 (J_2 J_3)$;
A_2), $(K_1 K_2) K_3 = K_1 (K_2 K_3)$;
A_3), $(K_1 J_1) J_2 = K_1 (J_1 J_2)$;
A_4), $(J_1 K_1) K_2 = J_1 (K_1 K_2)$;
A_5), $(J_1 K_1) J_2 = J_1 (K_1 J_2)$;
A_6), $(K_1 J_1) K_2 = K_1 (J_1 K_2)$;
A_7), $(J_1 J_2) K_1 = J_1 (J_2 K_1)$;
A_8), $(K_1 K_2) J_1 = K_1 (K_2 J_1)$;

C_1), E_j is closed under multiplication, that is $J_1 J_2 = J_3$;
C_2), E_k is closed under multiplication, that is $K_1 K_2 = K_3$;

C^j_{jk}), $J_1 K_1 = K_2 \quad (J_2 = 0)$;
C^k_{jk}), $J_1 K_1 = J_2 \quad (K_2 = 0)$;
C^j_{kj}), $K_1 J_1 = K_2 \quad (J_2 = 0)$;
C^k_{kj}), $K_1 J_1 = J_2 \quad (K_2 = 0)$;

C_r), not every X is a right-hand divisor of zero;
C_l), not every X is a left-hand divisor of zero;
C^j_r), not every J is a right-hand divisor of zero in E_j;
C^j_l), not every J is a left-hand divisor of zero in E_j;
C^k_r), not every K is a right-hand divisor of zero in E_k;
C^k_l), not every K is a left-hand divisor of zero in E_k.

§2.—*Dependencies.*

We prove the following dependencies:

D_1). The condition C_1 says that $J_1 J_2 = J_3$. Suppose that

$$J_1 J_2 = J_3 + K_3. \tag{3}$$

Multiplying both sides on the right by K

$$(J_1 J_2) K = J_3 K + K_3 K.$$

By A_7, C^j_{jk}, C^k_{jk} this becomes

$$0 = K_3 K. \tag{4}$$

By C^j_{kj}, C^k_{kj} $\quad 0 = K_3 J.$

Adding we obtain $\quad 0 = K_3 (J + K) = K_3 X.$

Hence by C_r $\quad K_3 = 0.$

The product $J_1 J_2$ being equal to J_3 proves the condition C_1.

TABLE I.

Notation.	Assumptions.						Consequence.	
D_1	A_7,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_r		C_1
D_2	A_3,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_l		C_1
D_3	A_7,	C^j_{jk},	C^k_{jk},			C^k_r		C_1
D_4	A_3,			C^j_{kj},	C^k_{kj},	C^k_l		C_1
D_5	A_8,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_r		C_2
D_6	A_4,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_l		C_2
D_7	A_8,			C^j_{kj},	C^k_{kj}	C^j_r		C_2
D_8	A_4,	C^j_{jk},	C^k_{jk},			C^j_l		C_2
D_9	A_5,			C^j_{kj},	C^k_{kj},	C^j_r	C^j_{jk}	
D_{10}	A_6,			C^j_{kj},	C^k_{kj},	C^k_l	C^k_{jk}	
D_{11}	A_6,	C^j_{jk},	C^k_{jk},			C^k_r	C^k_{kj}	
D_{12}	A_5,	C^j_{jk},	C^k_{jk},			C^j_l	C^j_{kj}	

D_2). Instead of multiplying (3) *on the right* by K and adding $K_1J = 0$, we multiply it *on the left* by K and add $JK_1 = 0$. In this way it is easily seen that C_1 is a consequence of A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, and C_l.

D_3). From equation (4) and C^k_r it follows that $K_1 = 0$. Hence C_1 is a consequence of A_7, C^j_{jk}, C^k_{jk}, C^k_r.

D_4). From the equation $KK_1 = 0$, which arises in the proof of D_2, it follows from C^k_l that $K_1 = 0$. Hence C_1 is a consequence of A_3, C^j_{kj}, C^k_{kj}, C^k_l.

D_5). This follows by interchanging j and k in D_1.

D_6). This follows by interchanging j and k in D_2.

D_7). This follows by interchanging j and k in D_3.

D_8). This follows by interchanging j and k in D_4.

D_9). The condition C^j_{jk} says that $J_1K_1 = K_2$. Suppose that

$$J_1K_1 = J_2 + K_2. \tag{5}$$

Multiplying both sides on the right by J

$$(J_1K_1)\,J = J_2J + K_2J.$$

By A_5, C^j_{kj}, C^k_{kj} this becomes

$$0 = J_2J. \tag{6}$$

By C^j_r we see that $J_2 = 0$. Therefore C^j_{jk} is a consequence of A_5, C^j_{kj}, C^k_{kj}, and C^j_r.

D_{10}). Instead of multiplying (5) *on the right* by J, we multiply it *on the left* by K. In this way it is readily seen that C^k_{jk} is a consequence of A_6, C^j_{kj}, C^k_{kj}, and C^k_l.

D_{11}). This follows by interchanging j and k in D_9.

D_{12}). This follows by interchanging j and k in D_{10}.

The conditions C_1, C_2, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, cannot be derived from any set of conditions other than the above.

§3.—*Definitions of Reducibility by Independent Assumptions.*

From Table I. it can be seen that there are seventy-eight different ways of defining the reducibility of a hypercomplex number system.

I.—The following eight definitions of reducibility are the only ones in which the division assumptions are on the set E as a whole.*

* All the definitions of this paper contain one or more division assumptions. There is only one definition, the classical one of Peirce-Scheffers, that does not contain a division assumption.

TABLE II.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_1	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_2	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	
R_3	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	....	C_1			
R_4	D_5	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_5	D_5	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	
R_6	D_5	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	....	C_1			
R_7	D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
	....	C_2			
R_8	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	
	....	C_2			

II.—The following thirty-eight definitions of reducibility are definitions in which the division assumptions are solely on the subsets E_j, E_k.

TABLE III.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_9	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_4	A_3, C^k_l	" "		C_1
R_{10}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
R_{11}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C_1			
R_{12}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_2
	D_4	A_3, C^k_l	" "		C_1
R_{13}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_2
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_{14}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C_2			
R_{15}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	....	C_1			
	....	C_2			
R_{16}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_8	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^k_{kj}			
R_{17}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C^k_{kj}			
	....	C_1			
R_{18}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_8	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^k_{kj}			
	....	C_2			
R_{19}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_1			
	....	C_2			
R_{20}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_8	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_{21}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C^j_{kj}			
	....	C_1			
R_{22}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
	....	C_2			
R_{23}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	....	C^j_{kj}			
	....	C_1			
	....	C_2			
R_{24}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, C^j_l	" "		C_2
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{25}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, C^j_l	" "		C_2
	D_8	A_7, C^k_r	" "		C_1
R_{26}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{27}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_3	A_7, C^k_r	" "		C_1
R_{28}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C_1			
R_{29}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C_2			
R_{30}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	....	C_1			
	....	C_2			
R_{31}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
R_{32}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^j_{jk}			
	....	C_1			

TABLE III.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{33}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
	....	C_2			
R_{34}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	....	C_1			
	....	C_2			
R_{35}	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^k_{jk}			
R_{36}	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^k_{jk}			
	....	C_1			
R_{37}	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^k_{jk}			
	....	C_2			
R_{38}	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	....	C^k_{jk}			
	....	C_1			
	....	C_2			
R_{39}	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
R_{40}	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
	....	C^k_{kj}			
R_{41}	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C^j_{kj}			
	....	C^k_{kj}			
	....	C_1			
R_{42}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_8, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
	....	C^k_{jk}			
R_{43}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
R_{44}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^j_{jk}			
	....	C^k_{jk}			
	....	C_1			
R_{45}	D_4	A_8, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
	....	C^k_{jk}			
	....	C_3			
R_{46}	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
	....	C^k_{kj}			
	....	C_3			

III.—The following thirty-two different ways of defining reducibility are based on assumptions concerning division in the set E and its subsets, E_j, E_k, simultaneously.

TABLE IV.*.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{47}	D_{12}	A_5, C^j_{jk}, C^{jk}_{k}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_3	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
R_{48}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_3	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
R_{49}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_3
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
R_{50}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, $*C^j_r$	" "		C_2
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
R_{51}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C_l	" "		C_3
	D_4	A_3, $*C^k_l$	" "		C_1

* A condition marked with a star is dependent on the other assumptions in the particular definition in which it appears. Thus $*C^j_r$ is dependent upon the other assumptions of R_{50} (§§4–5).

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.		Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{52}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},$	C^j_l	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},$	C^k_r	C^k_{kj}		
	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},$	C_l	" "		C_2
	D_3	$A_7,\ C^j_{jk},\ C^k_{jk},$	C^k_r		C_1	
R_{53}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},$	C^j_l	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},$	C^k_r	C^k_{kj}		
	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},$	C_l	" "		C_2
	D_3	$A_3,\ C^j_{jk},\ C^k_{jk},$	C_l	" "		C_1
R_{54}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},$	C^j_l	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},$	C^k_r	C^k_{kj}		
	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},$	C_l	" "		C_2
	D_1	$A_7,\ C^j_{jk},\ C^k_{jk},$	C_r	" "		C_1
R_{55}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},$	C^j_l	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},$	C^k_r	C^k_{kj}		
	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},$	C_r	" "		C_2
	D_4	$A_8,$	C^k_l	" "		C_1
R_{56}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},$	C^j_l	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},$	C^k_r	C^k_{kj}		
	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},$	C_r	" "		C_2
	D_8	$A_7,\ C^j_{jk},\ C^k_{jk},$	C^k_r		C_1	

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{57}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},\quad C^j_l$	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},\quad C^k_r$	C^k_{kj}		
	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},\quad C_r$	" "		C_2
	D_2	$A_3,\ C^j_{jk},\ C^k_{jk},\quad C_l$	" "		C_1
R_{58}	D_{12}	$A_5,\ C^j_{jk},\ C^k_{jk},\quad C^j_l$	C^j_{kj}		
	D_{11}	$A_6,\ C^j_{jk},\ C^k_{jk},\quad C^k_r$	C^k_{kj}		
	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},\quad C_r$	" "		C_2
	D_1	$A_7,\ C^j_{jk},\ C^k_{jk},\quad C_r$	" "		C_1
R_{59}	D_{10}	$A_6,\quad C^j_{kj},\ C^k_{kj},\quad C^k_l$	C^k_{jk}		
	D_9	$A_5,\quad C^j_{kj},\ C^k_{kj},\quad C^j_r$	C^j_{jk}		
	D_8	$A_4,\quad {}^*C^j_l$	" "		C_2
	D_3	$A_3,\quad C^j_{kj},\ C^k_{kj},\quad C_l$	" "		C_1
R_{60}	D_{10}	$A_6,\quad C^j_{kj},\ C^k_{kj},\quad C^k_l$	C^k_{jk}		
	D_9	$A_5,\quad C^j_{kj},\ C^k_{kj},\quad C^j_r$	C^j_{jk}		
	D_8	$A_4,\quad C^j_l$	" "		C_2
	D_1	$A_7,\quad C^j_{kj},\ C^k_{kj},\quad C_r$	" "		C_1
R_{61}	D_{10}	$A_6,\quad C^j_{kj},\ C^k_{kj},\quad C^k_l$	C^k_{jk}		
	D_9	$A_5,\quad C^j_{kj},\ C^k_{kj},\quad C^j_r$	C^j_{jk}		
	D_7	$A_8,\quad C^j_{kj},\ C^k_{kj},\quad C^j_r$		C_2	
	D_2	$A_3,\quad C^j_{kj},\ C^k_{kj},\quad C_l$	" "		C_1

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_{62}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1
R_{63}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{64}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_3	A_7, C^k_r	" "		C_1
R_{65}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_2	A_3, C^j_{kj}, C^k_{kj}, C_l	" "		C_1
R_{66}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{67}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_4	A_8, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{68}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_8	A_7, $*C^k_r$	" "		C_1
R_{69}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_2	A_8, C^j_{kj}, C^k_{kj}, C_l	" "		C_1
R_{70}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk},		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1
R_{71}	D_8	A_4, C^j_{jk}, C^k_{jk}, $*C^j_l$		C_2	
	D_2	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_{72}	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	

TABLE IV.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{73}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_2	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_{74}	D_7	A_8, C^j_{kj}, C^k_{kj}, $*C^j_r$		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	
R_{75}	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_4	A_8, C^j_{kj}, C^k_{kj}, $*C^k_l$		C_1	
R_{76}	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_8	A_7, C_{jk}, C^k_{jk}, C^k_r		C_1	
R_{77}	D_5	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_4	A_8, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{78}	D_5	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_8	A_7, C^j_{jk}, C^k_{jk}, $*C^k_r$		C_1	

§4.—*Independence Proofs.*

The sets of conditions $R_1, R_2, \ldots, R_{78}$ yield reducibility. It can be seen, however, that there are not enough conditions in any one of these sets to prove the associativity A of the system E. In each case it is necessary to add the conditions A_1, A_2. The set R_1 with the conditions A_1, A_2, adjoined, we designate by $R_1{}^A$ to indicate that the latter conditions yield both reducibility and associativity. Similarly $R_2{}^A \equiv [R_2, A_1, A_2]$; $\ldots$; $R_{78}^A \equiv [R_{78}, A_1, A_2]$.

Table V. contains proofs of the independence of the conditions in each of the sets $R_1, \ldots, R_{78}$, with the exception of eight—R_{50}, R_{51}, R_{59}, R_{68}, R_{71}, R_{74}, R_{75}, R_{78}—in which the subset division assumptions will be shown later to be redundant.

To prove these independencies we employ the following multiplication tables, for which the borders have been omitted:

I.

$$\begin{array}{ccc} e_2 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2$; $E_k = e_3$.

II.

$$\begin{array}{cc} e_1 + e_2 & 0 \\ 0 & e_2 \end{array}$$

$E_j = e_1$; $E_k = e_2$.

III.

$$\begin{array}{ccc} e_1 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2$; $E_k = e_3$.

IV.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1$; $E_k = e_2, e_3$.

V.

$$\begin{array}{cc} e_1 & e_1 \\ e_1 & e_2 \end{array}$$

$E_j = e_1$; $E_k = e_2$.

VI.

$$\begin{array}{ccc} 0 & 0 & e_1 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2$; $E_k = e_3$.

VII.

$$\begin{array}{ccc} e_1 & 0 & e_3 \\ 0 & e_2 & 0 \\ 0 & e_3 & 0 \end{array}$$

$E_j = e_1$; $E_k = e_2, e_3$.

VIII.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_2 & 0 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2$; $E_k = e_3$.

IX.

$$\begin{array}{ccc} 0 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2$; $E_k = e_3$.

TABLE V.—(*Independence Proofs*).

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	C_1	C_2	C^j_{jk}	C^k_{jk}	C^j_{kj}	C^k_{kj}	C_l	C_r	C^j_l	C^k_l	C^j_r	C^k_r	Proof.
1	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	I.
2	★	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (1).
3	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	★	II.
4	★	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (3).
5	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	★	III.
6	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	★	IV.
7	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	Interchange j, k in (6).
8	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	Interchange j, k in (5).
9	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	★	V.
10	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	VI.
11	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	Interchange j, k in (9).
12	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	VII.
13	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	★	★	★	Interchange j, k in (12).
14	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	Interchange j, k in (10).
15	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	VIII.
16	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	★	Interchange j, k in (15).
17	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	IX.
18	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	Interchange j, k in (17).

According to $R_1{}^4$ the system E is reducible and associative under the assumptions A_1, A_2, A_3, A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l.

That A_1 is independent of the other conditions in $R_1{}^4$ is shown in row 1 of Table V.

The independence of A_2 is shown in row 2.

Similarly the independence of any one of the conditions of the others can be seen by looking down its column in Table V. for the square marked i. In some cases, as in column A_5, the letter i occurs in both row 5 and row 6. In order to prove A_5 independent of the other conditions in $R_2{}^4$ we use row 6; but

to prove A_5 independent of the other conditions in R^4_{24}, row 5 must be used. Likewise there can be no confusion in similar cases—such for example as in row 7 and row 8.

§5.—*Dependence proofs* (R_{50}, R_{51}, R_{59}, R_{68}, R_{71}, R_{74}, R_{75}, R_{78}).

1. In R_{50}* the assumptions are not independent. We proceed to prove that C^j_r *is a consequence of the others*, which are mutually independent.

We know by Table I., D_{12}, D_{11}, that C^j_{kj} and C^k_{kj} are consequences of the assumptions of R_{50} (C^j_r being omitted). Table I. D_1, D_5 show that C_1 and C_2 are consequences of A_7, A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r, and, therefore, can be derived from the above-mentioned independent conditions of R_{50}. According to the condition C_r, there exist a J and a K such that

$$(J_1 + K_1)(J + K) = 0, \text{ only if } J_1 = 0 = K_1.$$

Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, that there exist a J and a K such that

$$J_1J + K_1K = 0, \text{ only if } J_1 = 0 = K_1.$$

By C_1, C_2 this may be written $J_3 + K_3 = 0$. Multiplying the last equation on the left by J' it appears that there exist a J and a K such that

$$J'J_3 + J'K_3 = 0, \text{ only if } J_1 = 0 = K_1.$$

Since $J'K_3 = 0$ (by C^j_{jk}, C^k_{jk}) we conclude there exists a J such that

$$J'J_3 = J'(J_1J) = 0, \text{ only if } J_1 = 0.$$

From C^j_l it follows that there exists a J such that

$$J_1J = 0, \text{ only if } J_1 = 0.$$

2. That C^k_l is dependent upon the remaining assumptions of R_{51} can be shown in a similar manner.†

3. That C^j_l is dependent upon the remaining assumptions of R_{59} is seen by interchanging k and j in 2 (i. e. in R_{51}).

4. That C^k_r is dependent upon the remaining assumptions of R_{68} is seen by interchanging j and k in 1 (i. e. in R_{50}).

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* Also in R^4_{50}.

† The following proofs for 2, 3, 4 may also be employed. In R_{50} change *left* to *right*, obtaining thus R_{50}. In R_{50} interchange j and k, obtaining thus R_{51}.

5. In R_{71}, C^j_l is dependent upon the remaining assumptions. For, (Table I.) D_2, D_6 show that C_1 and C_2 are consequences of A_3, A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l (cf. R_1). C_l means that there exist a J and a K such that

$$(J+K)(J_1+K_1)=0, \text{ only if } J_1=0=K_1.$$

Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, that

$$JJ_1+KK_1=0, \text{ only if } J_1=0=K_1.$$

Multiplying on the left by J', in view of A_4, C^j_{jk}, C^k_{jk}, it appears that there exists a J such that

$$J'(JJ_1)=0, \text{ only if } J_1=0,$$

(or, only if $JJ_1=0$). This is C^j_l.

6. That C^j_r is dependent upon the remaining assumptions of R_{74} (cf. R_5) is seen by changing *left* to *right* in R_{71}.

7. That C^k_l is dependent upon the remaining assumptions of R_{75} is seen by interchanging j and k in R_{71} (cf. R_1).

8. That C^k_r is dependent upon the remaining assumptions of R_{76} is seen by interchanging j and k in R_{74} (cf. R_5).

§6.—*Semireducibility.*

An associative hypercomplex number system E, containing a modulus, is said to be *semireducible of the first kind* (Transactions, vol. 4 (October, 1903), pp. 437-444) when the following conditions are fulfilled:

$$A,\ C^j_{jk},\ C^j_{kj},\ C_1,\ C_2,\ C_r,\ C_l. \tag{7}$$

In this body of conditions, each one is independent of all the others.* In

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* At the time this definition was framed I was not aware that a more general case had been studied by Molien in the Mathematische Annalen, vol. 41 (1893), pp. 92-93. When E satisfies the above conditions with C_1 omitted, it has according to Molien an *accompanying system*. Professor E. H. Moore has recently pointed out to me that when the conditions

$$A,\ C^j_{kj},\ C_2,\ C_r,\ C_l \tag{7a}$$

are satisfied, the group G of the system E will be reducible and will take the form

$$x'_{j_3}=\sum_{j_1}\{\sum_{j_2}\gamma_{j_1j_2j_3}\,y_{j_2}+\sum_{k_2}\gamma_{j_1k_2j_3}\,y_{k_2}\}x_{j_1},$$

$$x'_{k_3}=\sum_{j_1}\{\sum_{j_2}\gamma_{j_1j_2k_3}\,y_{j_2}+\sum_{k_2}\gamma_{j_1k_2k_3}\,y_{k_2}\}x_{j_1}+\sum_{k_1}\{\sum_{j_2}\gamma_{k_1j_2k_3}\,y_{j_2}+\sum_{k_2}\gamma_{k_1k_2k_3}\,y_{k_2}\}x_{k_1},$$

$$(j=1.\,.\,.\,.\,.\,m;\ k=m+1.\,.\,.\,.\,n).$$

The above-mentioned definition of Molien is the special case obtained when $\gamma_{jkj}=0$ (C^j_{jk}). My definition of semireducibility of the first kind is the special case obtained from Molien's "nicht ursprüngliche" Systems when $\gamma_{jjk}=0$ (C_1). Since the conditions (7) are mutually independent, the Molien conditions, which are obtained by omitting C_1 from (7) must be independent and the conditions (7a) which are obtained from Molien's by omitting C^j_{jk} must evidently be independent also.

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order to make the independence proofs we add the following systems to those of §4.

X.

e_1	e_2	e_3
e_2	e_3	0
e_3	0	0

$E_j = e_1, e_2;\ E_k = e_3.$

XI.

e_1	e_2	e_3
e_2	e_1	e_3
e_3	$-e_3$	0

$E_j = e_1;\ E_k = e_2, e_3.$

A	$C^{\prime}_{jk}$	$C^{\prime}_{kj}$	C_1	C_2	C_r	C_l	Proof.
i	★	★	★	★	★	★	I.
★	i	★	★	★	★	★	VI.
★	★	i	★	★	★	★	VII. interchanging j and k.
★	★	★	i	★	★	★	X.
★	★	★	★	i	★	★	XI.
★	★	★	★	★	i	★	IX.
★	★	★	★	★	★	i	VIII.

§7.

It was shown in §1 that a hypercomplex number system is reducible when by a proper choice of the units it can be brought to the form

$$E \equiv E_j E_k = e_1 \ldots . e_m e_{m+1} \ldots . e_n,$$

where the following conditions are fulfilled:

A), associativity;

C_{jk}), $e_j e_k = 0$, i. e. every $\gamma_{jki} = 0$;

C_{kj}), $e_k e_j = 0$, i. e. every $\gamma_{kji} = 0$;

C_r), right-hand division possible; or

C_l), left-hand division possible.

In other words, the conditions

$$C_1), \qquad e_{j_1}e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{jjk} = 0),$$

$$C_2), \qquad e_{k_1}e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{kkj} = 0),$$

are both consequences of A, C_{jk}, C_{kj}, C_r (or C_l).

Suppose now that C_{jk}, C_{kj} are fulfilled, but C_l and C_r are not; in this case we have no reason to conclude that every γ_{jjk} and every γ_{kkj} is zero.* Nevertheless it can be demonstrated that there exist two hypercomplex number systems

$$F_j \equiv f_1 \; \ldots . \; f_m, \; F_k = f_{m+1}, \ldots . , f_n,$$

such that

$$f_{j_1}f_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} f_{j_3}, \; f_{k_1}f_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} f_{k_3}.$$

In particular, when either C_l on C_r is fulfilled, we obtain $f_i = e_i$ (i $= 1, \ldots . , n$).

In proof we have from the associativity condition

$$(e_{j_1} e_{j_2})\, e_{j_4} = e_{j_1} (e_{j_2} e_{j_4}),$$

$$\sum_{i_3=1}^{n} (\gamma_{j_1 j_2 i_3}\gamma_{i_3 j_4 i_5} - \gamma_{j_2 j_4 i_3}\gamma_{j_1 i_3 i_5}) = 0, \tag{8}$$

$$(j_1, j_2, j_4 = 1, \ldots . , m; \; i_5 = 1, \ldots . , n).$$

By C_{jk} and C_{kj} (7) becomes

$$\sum_{j_3=1}^{m} (\gamma_{j_1 j_2 j_3}\gamma_{j_3 j_4 i_5} - \gamma_{j_2 j_4 j_3}\gamma_{j_1 j_3 i_5}) = 0, \tag{9}$$

$$(j_1, j_2, j_4 = 1, \ldots . , m; \; i_5 = 1, \ldots . , n).$$

---

* As an example of this possibility consider the system $\begin{matrix} e_1 + e_2 & 0 \\ 0 & 0. \end{matrix}$ It can easily be verified that the conditions A, C_{jk}, C_{kj} are fulfilled, but clearly $\gamma_{112} = 1 \; (\neq 0)$.

The m^3n equations (8) include the m^4 equations

$$\sum_{j_3=1}^{m} (\gamma_{j_1j_2j_3}\gamma_{j_3j_4j_5} - \gamma_{j_2j_4j_3}\gamma_{j_1j_3j_5}) = 0, \tag{10}$$

$$(j_1, j_2, j_4, j_5 = 1, \ldots, m),$$

and this shows that there exists a system in m units

$$F_j = f_1 \ldots f_m \text{ such that } f_{j_1}f_{j_2} = \sum_{j_3} \gamma_{j_1j_2j_3}f_{j_3}.$$

By interchanging j and k it follows that there exists a system in $m - n$ units

$$F_k = f_{m+1} \ldots f_n \text{ such that } f_{k_1}f_{k_2} = \sum_{k_3} \gamma_{k_1k_2k_3}f_{k_3}.$$

THE UNIVERSITY OF CHICAGO, *June*, 1904.

Quintic Curves for which $P = 1$.[*]

By Peter Field.

The general equation of a quintic curve with five fixed double points contains five arbitrary constants. If, therefore, the curve is required to pass through five additional points, it will be completely determined.

Let ϕ_0 and ϕ_1 be two conics through the points $1, 2, 3, 4$, U_0 and U_1 nodal cubics through the same four points and having a node at a point 5. Then the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$, where λ is an arbitrary constant, represents a quintic curve having nodes at the five given points. Moreover this equation is general. For, suppose $S = 0$ to be the equation of any quintic curve having the five given double points. The curve $S = 0$ intersects the conic $\phi_0 = 0$ in a pair of points (aside from the nodal points). Take U_1 through these points. Similarly pass U_0 through the two points of intersection (i. e. the two points which are not nodal points on S) of S and ϕ_1 and determine λ so that the coordinates of an additional point on $S = 0$ satisfy the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$. The only restriction in the selection of this last point is that it must not be the twenty-fifth point of intersection of two quintic curves having $1, 2, 3, 4, 5$ as nodal points and the four points of $S\phi_0$ and $S\phi_1$, which are ordinary points on S, as ordinary points. The equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ must then be identical with the equation $S = 0$.

Just as in the case of the unicursal curves, if λ is small, the curve is very nearly of the form of $\phi_0 U_0$ with the exception of two breaks at the two intersections which do not lie on the curve. Four combinations of these two breaks are possible, and it is easily seen that all of them are permissible; for (Fig. 1) the way in which the break is made at 6 can be changed by simply changing the sign of

[*] For a partial list of the forms of the non-singular quintic curves, see Bancroft, American Journal of Mathematics, Vol. X.

λ. Suppose now that the generating point moves from 6 toward 7 following very closely to ϕ_0. The way in which the break at 7 is made will depend on which side of ϕ_0 the generating point approaches 7, but this is arbitrary as the quintic crosses ϕ_0, aside from the nodal points, only at the two remaining points of intersection of U_1 and ϕ_0, and these are arbitrary. The equation might equally well have been put in a variety of other forms.

As in the case of the unicursals, a few additional forms are given by considering also the case of a quartic and a straight line, and for the same reason, viz. that while the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ is general, we obtain only the forms for small values of λ and do not obtain the transitional forms. This case,

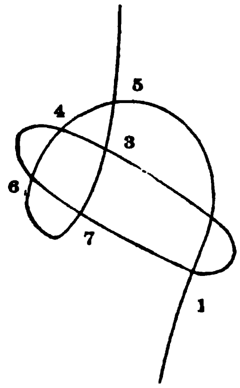


Fig. 1.

however, differs from the case of the unicursals in that there is no difficulty in writing the equation in the form $a_0 U_0 - \lambda a_1 U_1 = 0$ where the a's are linear and the U's quartic functions in the variables.

The method that has been used for determining the forms of quintic curves having five crunodes may by a slight modification, be used for determining the forms of curves having one or more cusps. For one cusp $\phi_0 U_0$ can be taken with two consecutive intersections and one of the breaks made at one of these points. For two cusps $\phi_0 U_0$ can be taken with an additional pair of consecutive intersections and the second break made at this point; for three cusps the preceding only needs to be modified by taking U_0 a cuspidal cubic. In case four cusps are desired, U_0 can be taken a cuspidal cubic and $\phi_0 U_0$ with three consecutive intersections at one point and two at another, the breaks now being made at the second of the three consecutive points and at one of the two consecutive points.

If U_0 is a cuspidal cubic and U_0 and ϕ_0 have three consecutive intersections at two different points and the breaks are made in each case at the second of the three points, a quintic curve with five cusps is obtained. Two pairs of these cusps have their vertices approaching coincidence, while the above curve, with four cusps, has one such pair. In each case the breaks are, of course, supposed to be made in such a way as to give cusps.*

The equation of a quintic curve having four cusps, can be written down at once as follows: Let α, α_1 be the lines and ϕ, ϕ_1 the conics represented in the accompanying figure, then the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ represents a quintic

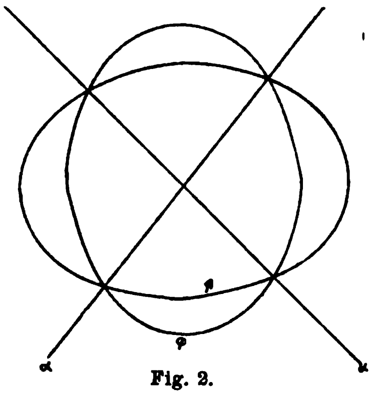


Fig. 2.

curve having cusps at the four points $\phi\phi_1$. The equation also contains a sufficient number of constants to fix another double point. Each of the conics passes through the four cuspidal points and, in addition, is tangent to the quintic at two of these points. The above equation, therefore, does not represent the general quintic curve with four cusps, as no such conics can be drawn in the general case.

The acnodal forms might also have been examined as in every case where a cusp is admissible; an acnode is also possible.†

Since the curves are not all unipartite, the sequence of the double points can not be used as a basis of classification‡ in the same way as was done for the

* The justification for making a break at a specified one of the consecutive points, rests on making the break while the points are distinct, and then considering them as neighboring points.

† See R. Gentry, On the Forms of Plane Quartic Curves, pp. 27, dissertation, Bryn Mawr, 1896.

‡ For an explanation of this method, see Meyer, Anwendungen der Topologie auf die Gestalten der Algebraischen Curven, Muenchen, 1878; or Tait, Edinburg Transactions, 1876-77.

unicursals,* but this scheme will be modified as follows: curves which are unipartite or which have all the nodes on one circuit, will be regarded as similar in case they have the same sequence of double points. In other cases, they will be regarded as similar in case the two branches of one curve have the same sequence of double points as those of another, as for instance 2a and 2c. In case curves, which are classed the same, differ considerably in appearance, several figures will be given.

Fig. 33 is worthy of special notice, it being the only form which permits five cusps. It has twenty real bitangents and five real inflexions. In case the loops are replaced by cusps, it has no bitangents, but it still has five real inflexions.

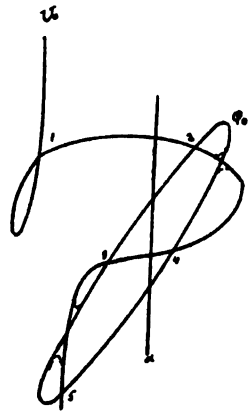


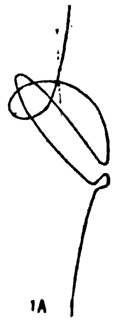
Fig. 3.

The equation of such a curve can be written down at once. Let ϕ_0, U_0, α be the curves designated in the figure; also, let ϕ_1 be a conic through the points 1, 2, 3, 4, 5 and λ a small constant, then the equation $\phi_0 U_0 - \lambda \alpha \phi_1^2 = 0$ represents a curve of the form given in Fig. 33. The sign of λ must be taken so that the breaks occur in the way indicated in the figure.

A table giving the sequence of the double points for the various forms is added. In case the curve is bipartite, the sequence of the double points for the two parts is separated by a dash.

* American Journal of Mathematics, Vol. XXVI (1904).

PLATE I.



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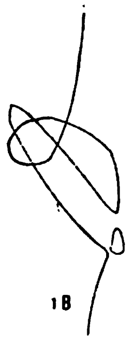

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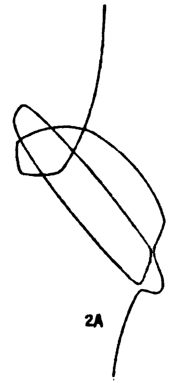
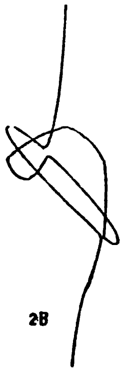


2A



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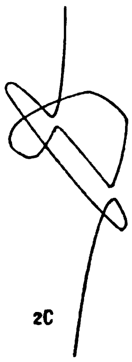

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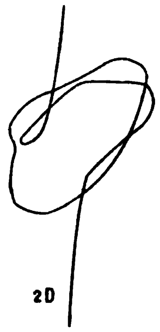
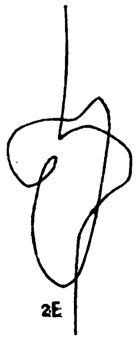


2D



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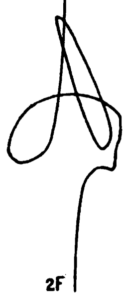

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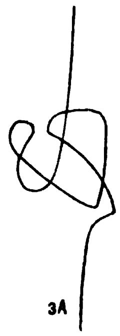

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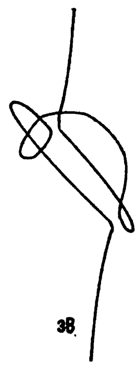

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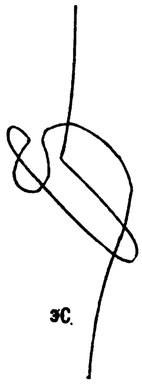

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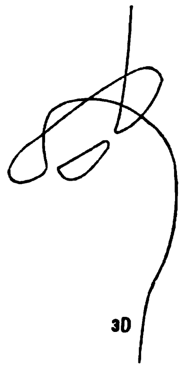
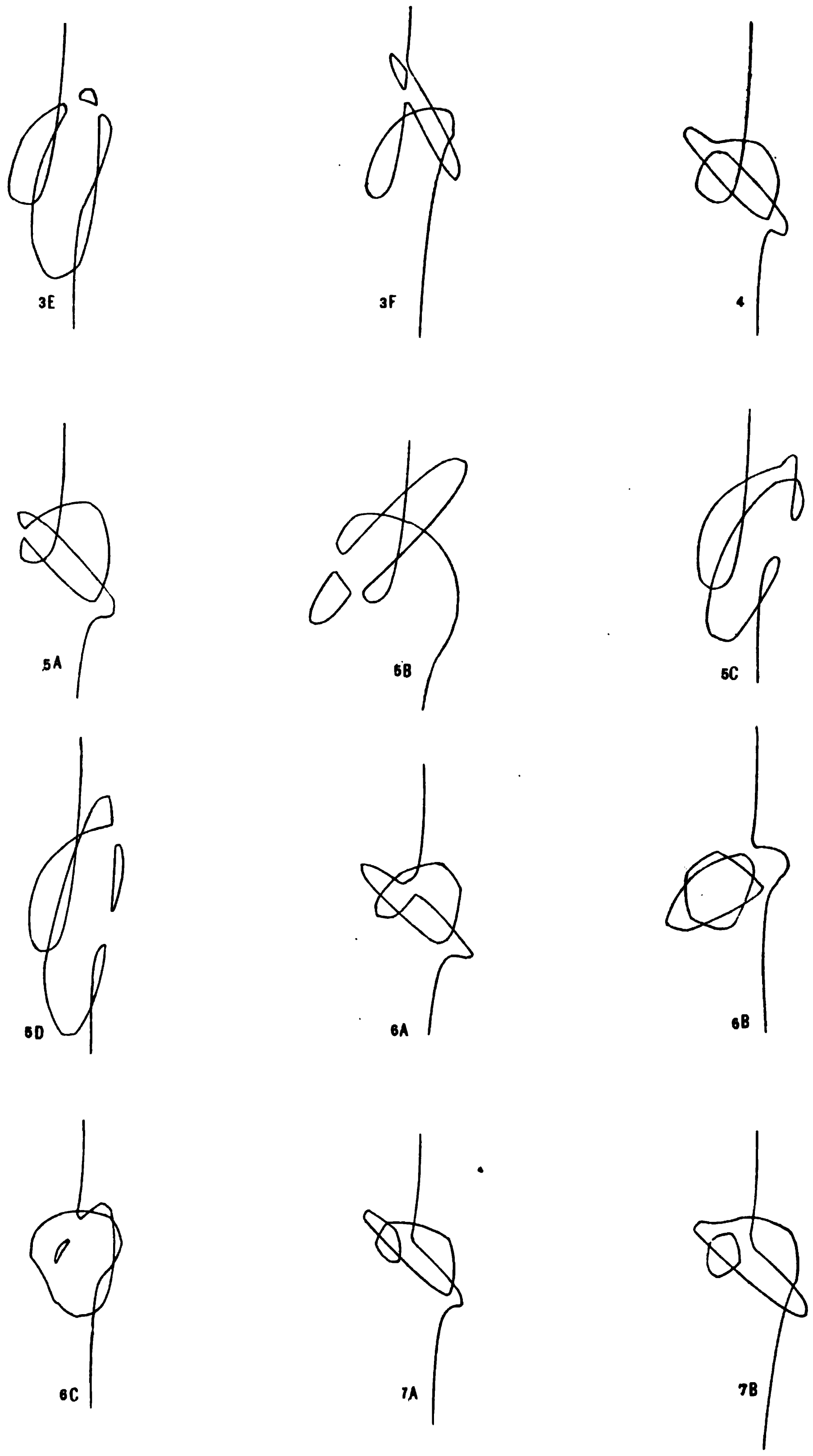


3D

PLATE II.



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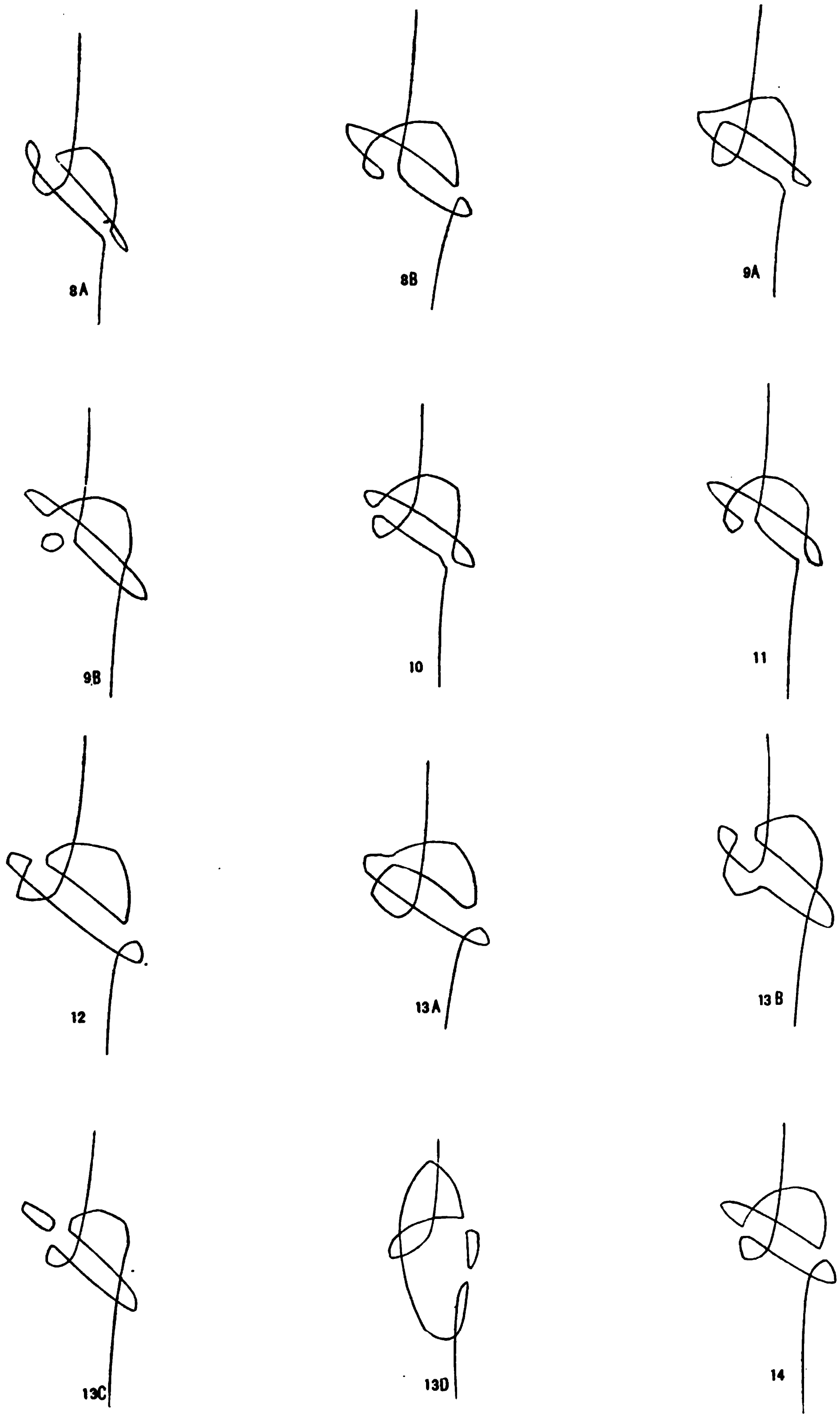

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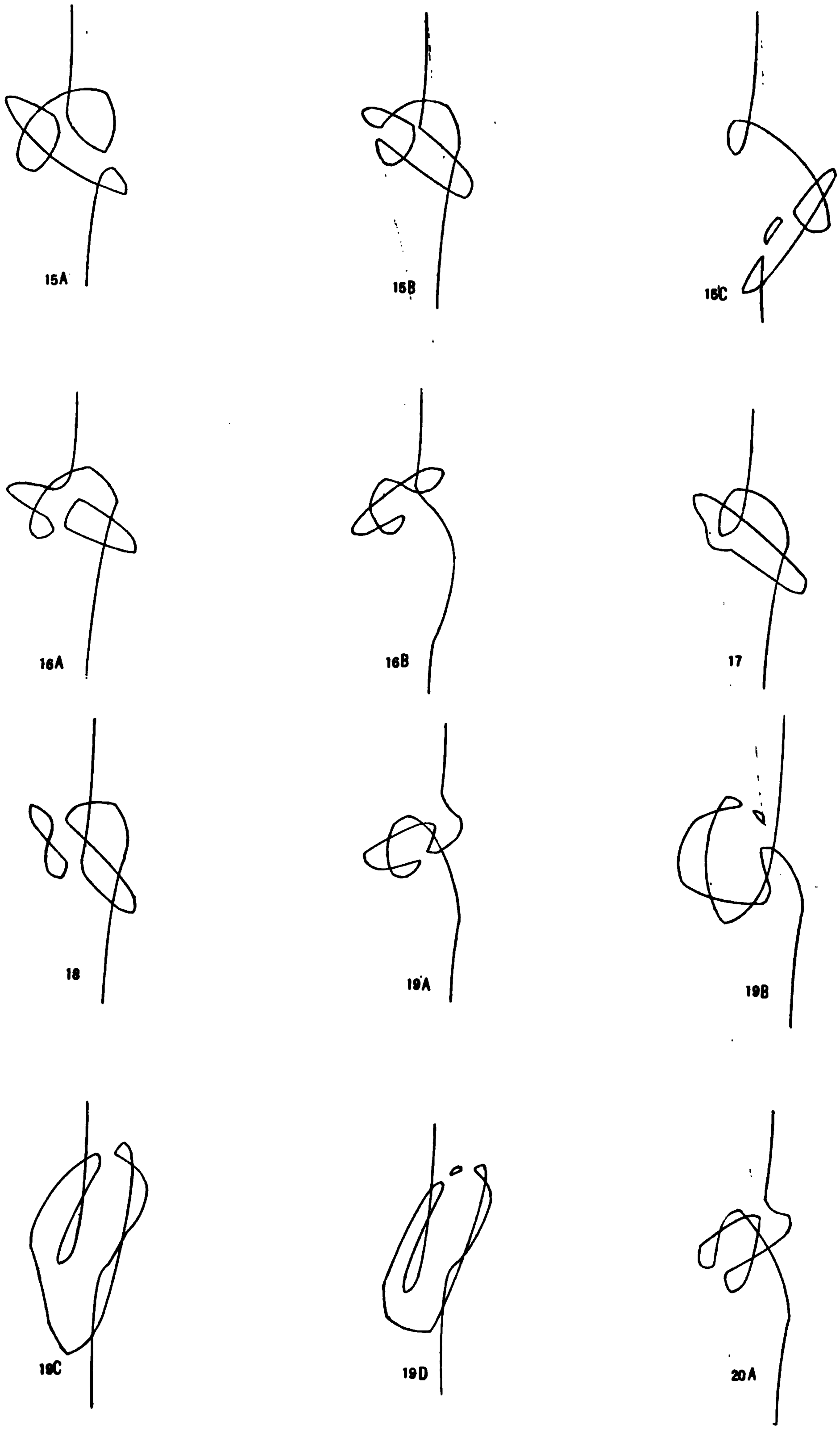

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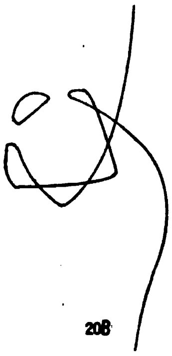
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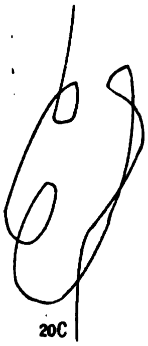
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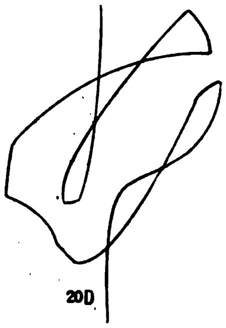
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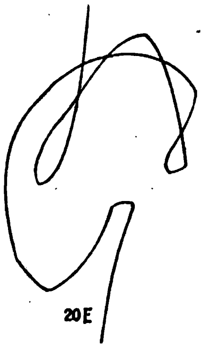
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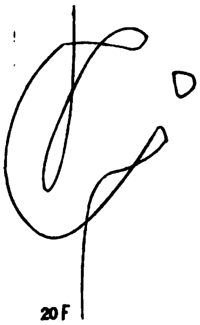
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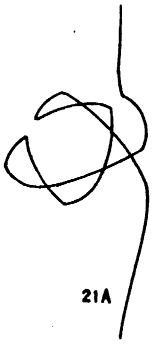
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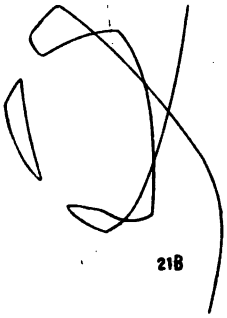
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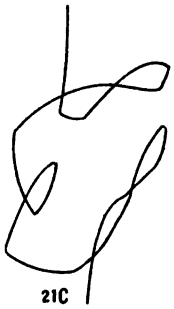
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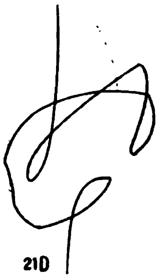
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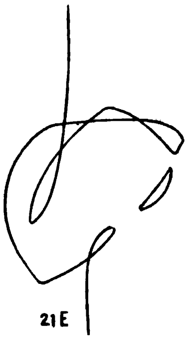
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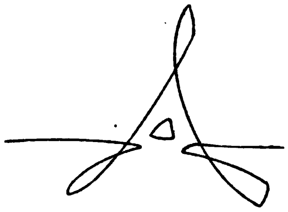
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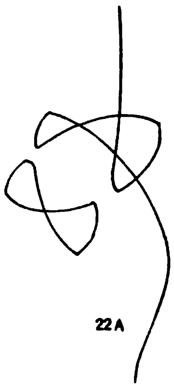
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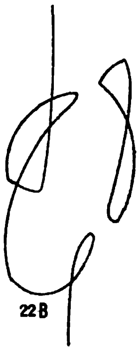
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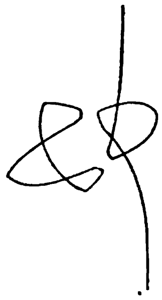
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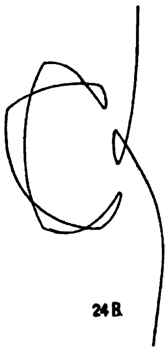
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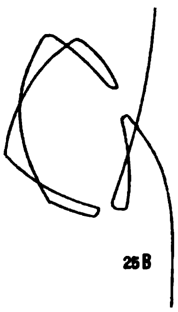
24C


25A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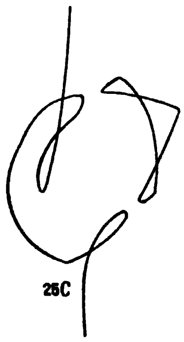
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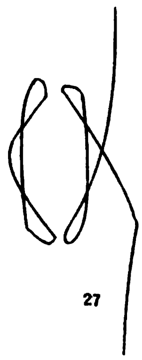
26A


26B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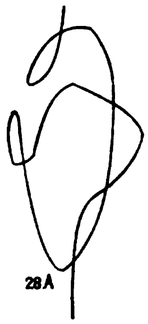
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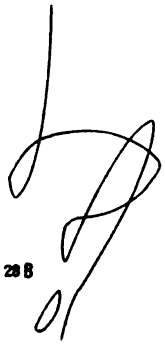
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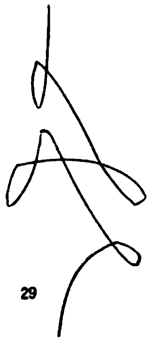
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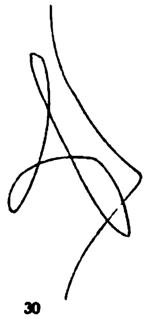
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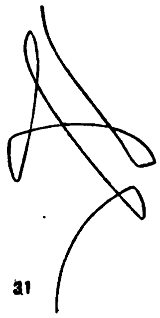
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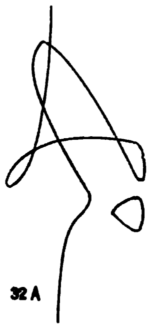
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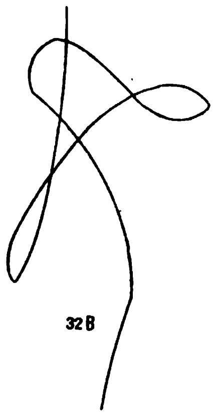
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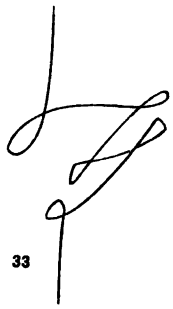
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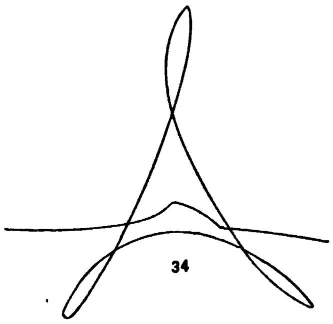
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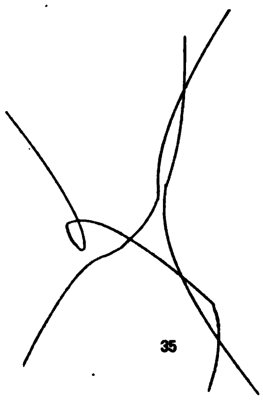
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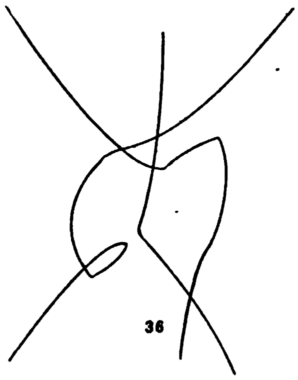
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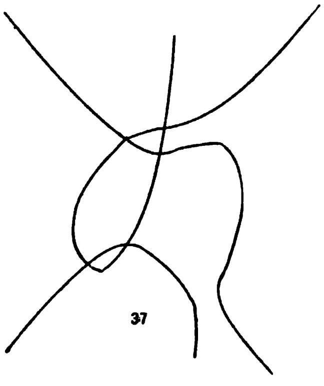
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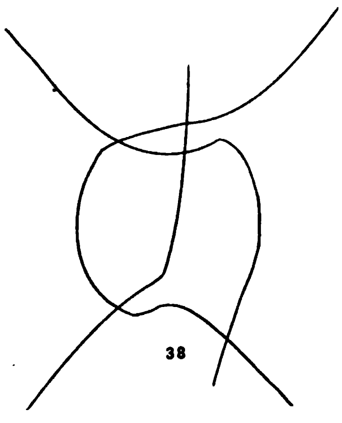
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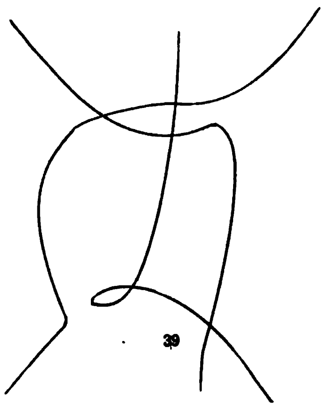
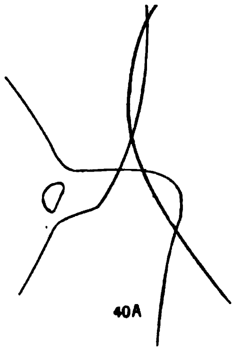
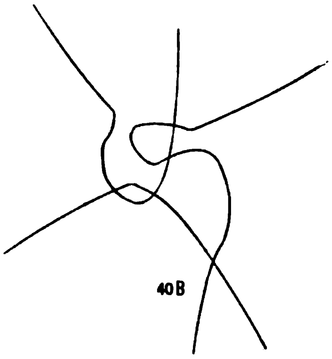

39

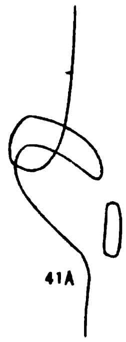
PLATE VIII.



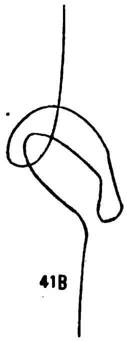
40A



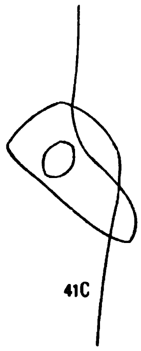
40B



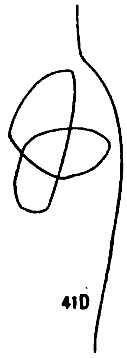
41A



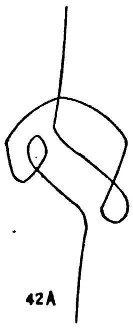
41B



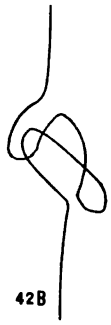
41C



41D



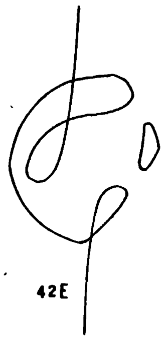
42A



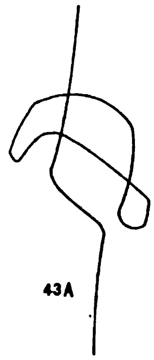
42B

42C

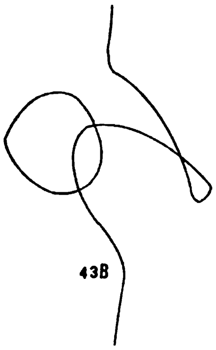
42D

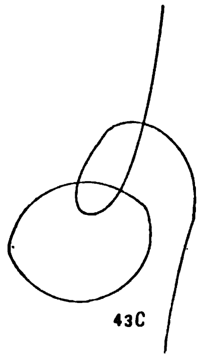


42E

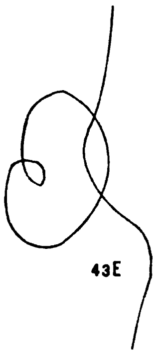


43A


43B


43C


43D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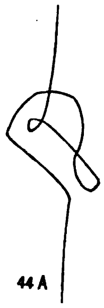
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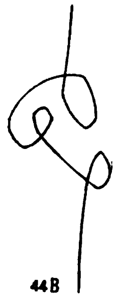
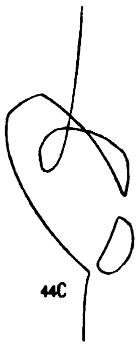

44B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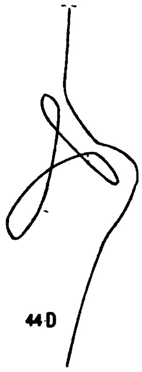
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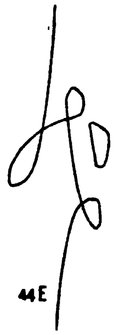
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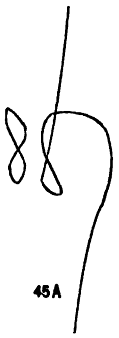
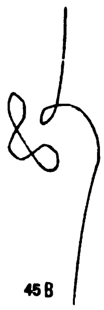
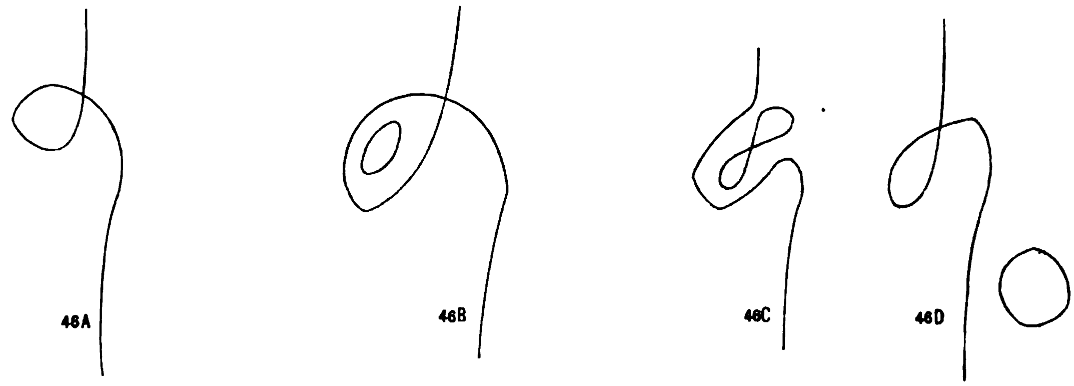

45A


45B


45C

PLATE X.



46A 46B 46C 46D

Forms 41–45 have two imaginary double points; 46 has four.

1	*abcdeabedc.*	24	*aabcddcb — ee.*
2	*abcdea — bcde.*	25	*aabccb — ddee.*
3	*aabcdecdeb.*	26	*aabccbdd — ee.*
4	*abcdec — abed.*	27	*aabbcc — ddee.*
5	*aabcddebce.*	28	*aabcdeebcd.*
6	*abcdeabcde.*	29	*aabbccdeed.*
7	*abcabcde — de.*	30	*aabbcdec — de.*
8	*aabcdb — eecd.*	31	*aabccddbee.*
9	*aabcdbedce.*	32	*aabccdebde.*
10	*aabccd — eebd.*	33	*aabbccddee.*
11	*aabcddce — eb.*	34	*aabbcdde — ce.*
12	*aabcdeeb — cd.*	35	*aabcdecd — be.*
13	*aabcdecbed.*	36	*bedced — aabc.*
14	*aabbcd — eecd.*	37	*abcdeadc — eb.*
15	*aabcdbcdee.*	38	*abcadcedbe.*
16	*abccbade — de.*	39	*aabcdbed — ce.*
17	*abcedc — abed.*	40	*abcdecdabe.*
18	*abcdbadc — ee.*	41	*abcabc.*
19	*aabcdeedcb.*	42	*aabccb.*
20	*aabcddeecb.*	43	*aabbcc.*
21	*aabccbdeed.*	44	*aabc — bc.*
22	*aabcdbcd — ee.*	45	*aabb — cc.*
23	*abcabc — ddee.*	46	*aa.*

UNIVERSITY OF MICHIGAN, *October*, 1903.

Classification of the Surfaces of Singularities of the Quadratic Spherical Complex.

By C. L. E. Moore.

In Vol. 1, page 381, of the Transactions of the American Mathematical Society, Professor P. F. Smith has discussed the surface of singularities of the general quadratic spherical complex. It is the aim of this paper to complete the classification of these surfaces as Weiler* has done for the quadratic line complex. Since spheres as well as lines can be represented by six homogeneous coordinates, Weiler's symbols and notations for the fundamental complexes are used, but no further use is made of line geometry. The properties of those surfaces which can be obtained from cyclides are investigated by means of the transformation used by Smith, and his transformation notation is adopted.

(I).

First Canonical Form. [111111]

1. $$\Pi = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0,$$
$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 = 0.$$

This is the surface discussed by Smith. He showed that the surface has six double sphero-quartics. The 32 minimum lines are also double, for $(A) = (DID)$ and (D) transforms a minimum line on a surface into two double minimum lines on the transformed surface,† (I) leaves the lines double. Hence the 16 minimum lines of the cyclide by (A) transform into 32 double lines of the transformed surface.

* Math. Ann., Vol. VII, page 145.

† See Roberts, "On Parallel Surfaces," Proceedings of the London Mathematical Society, Vol. IV, page 233.

These lines are arranged so that each line g is cut by six others, for a minimum line on the cyclide transforms into two intersecting lines on the transformed surface, and two intersecting minimum lines transform into two pairs of intersecting lines. Hence g and the five lines of the cyclide which cut it transform into two intersecting lines and five other lines cutting each one.

Snyder* has shown that the surface of singularities of F is also the surface of singularities for the ∞^1 complexes obtained by giving K all real values in

$$\frac{x_1^2}{k+a_1}+\frac{x_2^2}{k+a_2}+\frac{x_3^2}{k+a_3}+\frac{x_4^2}{k+a_4}+\frac{x_5^2}{k+a_5}+\frac{x_6^2}{k+a_6}=0,$$

and that, consequently, the surface belongs to the congruences

$$x_k=0,\quad \sum_1^6 \frac{x_i^2}{a_i-a_k}=0,\qquad i \neq k.$$

2. [(11) 1111].

$$\frac{x_1^2+x_2^2}{k+a_1}+\frac{x_3^2}{k+a_3}+\frac{x_4^2}{k+a_4}+\frac{x_5^2}{k+a_5}+\frac{x_6^2}{k+a_6}=0.$$

In 1, if $x_1=0$ is the complex of points, the surface 111111 becomes the general cyclide.† The generators which belong to $x_2=0$ have double contact with the surface. Hence [(11) 1111] becomes the points of $x_2=0$ which have double contact with the surface, i. e. the focal line (general sphero-quartic, c_4) which lies on the fundamental sphere of $x_2=0$.

By a general transformation (A), the complex $x_1=0$ inverts into a general complex and the focal line c_4 inverts into the surface of singularities of the surface [(11) 1111]. c_4 lies on a non-directed sphere s, therefore, its points invert into spheres which touch the two director spheres into which s inverts; furthermore, c_4 cuts each generator of s in two points which invert into spheres having a line in common with one of the directrices. Hence, *the surface has two double directrices.* It is of order and class 16, passes 8 times through K, the imaginary circle at infinity, the locus of centers is a quartic curve. The lines of curvature‡ are spherical curves of order 16. There are two special ones of order 8 which

* Bulletin of the American Mathematical Society, Vol. 4, page 152.

† See Loria, "Ricerche intro alla Geometria Della Sfera," Memorie di Torino, Vol. 36, ser. 2, 1884, page 75.

‡ Snyder, "Lines of Curvature," etc., American Journal, Vol. XXII, page 96.

are the locus of the points of contact of the generators with each of the directrices. From (2) we see that the surface may be generated in four ways as the envelope of ∞^2 spheres, hence it has four developables of bitangent planes, and for the generation as the envelope of ∞^1 spheres, the developable degrades into four planes. c_4 cuts the fundamental sphere of A in four points, hence the transformed surface has four nodes which lie on a circle.

The complex A may be so chosen that the director spheres may be, (1) two spheres, (2) two points, (3) point and sphere, (4) point and plane, (5) plane and sphere, (6) two planes. In the last case, the lines of curvature and locus of centers are plane curves. The directrices cannot have united position.

If c_4 lies on the sphere s', whose points invert into planes by (A), the surface becomes a developable circumscribed to a sphere along a sphero-quartic. In this case, the special complex which has s' for fundamental sphere, transforms into the complex of planes, i. e. one of the special complexes belonging to the congruence $x_1 + kx_2 = 0$, is the complex of planes, and since x_3, x_4, x_5, x_6 are in involution with it, they are plane complexes (fundamental sphere is a plane). The surface then belongs to that class of surfaces discussed by Smith in the Annals,* which we shall call *Laguerre surfaces*.† It is the Laguerre transform of the quadric cone as may be seen by transforming x_6 into the complex of points. The fundamental complexes become

$$x_1 + ix_2 = \nu_1; \quad x_1 - ix_2 = \mu_1; \quad x_3, x_4, x_5, x_6 = \xi, \eta, \zeta, \lambda,$$

and the surface is seen to be a cone.‡

By a general (E), the cone becomes the surface described above. It is of order 8, class 4, the characteristics of the cuspidal curve are $m = 12$, $r = 8$, $n = 4$, $d = 0$, etc. The surface does not contain K. Spheres concentric with the director sphere cut the surface in lines of curvature,§ hence the lines of curvature are of order 16.

The cone has three planes of symmetry, therefore, it is sibireciprocal under

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* Annals of Mathematics, ser. 2, Vol. 1, page 153.

† Laguerre surfaces may be defined as the envelope of planes belonging to a quadratic spherical complex. They may be derived from the quadric surfaces by means of an inversion in a plane complex.

‡ The coordinates ξ, η, ζ, λ, μ, ν are those used by Snyder in "Criteria for Nodes in Dupin's Cyclides," Annals of Mathematics, Vol. 2, June, 1897.

§ Snyder, "Lines of Curvature on Annular Surfaces," etc., American Journal of Mathematics, Vol. XXII, page 96.

three inversions and also λ (the complex of points), hence the transformed surface is sibireciprocal under four inversions, and, consequently, has *four double conics.* This is the complete double curve of the developable.

3. $[(11)(11)\,11]$. $\quad a_1 = a_2, \quad a_3 = a_4,$

$$\frac{x_1^2 + x_2^2}{k + a_1} + \frac{x_3^2 + x_4^2}{k + a_3} + \frac{x_5^2}{k + a_5} + \frac{x_6^2}{k + a_6} = 0.$$

In 2, if $x_3 = 0$ is the complex of points, the surface becomes the binodal cyclide, then $[(11)(11)\,11]$ is the focal line on the fundamental sphere of x_4 which, in this case, breaks up into two circles. By a transformation (A), the surface becomes two Dupin cyclides. They have in common two spheres of each generation, viz. the transform of the non-directed sphere on which the circles lie and the transform of their common points. The cyclides are so related that two nodes of one and two nodes of the corresponding generation of the other lie on a circle.

If the sphere on which the circles lie is the sphere whose points transform into planes, the surface becomes two cones of revolution which have two common tangent planes.

4. $[(11)(11)(11)]$.

$$a_1 = a_2, \quad a_3 = a_4, \quad a_5 = a_6.$$

In 3, if $x_1 = 0$ is the complex of points, the focal line on the fundamental sphere of x_2 breaks up into four minimum lines, therefore, the general surface consists of two Dupin cyclides which degrade into two skew quadrilaterals so related that each line of the first cuts one line of the second. These lines can be arranged into two groups of four non-intersecting lines such that any line of one cuts three lines of the other.

For the Laguerre surface, each cyclide becomes two finite minimum lines and* the line at infinity in their plane counted twice.

5. $[(111)\,111]$. $\quad a_1 = a_2 = a_3,$

$$\frac{x_1^2 + x_2^2 + x_3^2}{k + a_1} + \frac{x_4^2}{k + a_4} + \frac{x_5^2}{k + a_5} + \frac{x_6^2}{k + a_6} = 0.$$

* By finite minimum line is meant a minimum line not lying in plane at infinity.

If $k = -a_1$, we have

$$x_1 = x_2 = x_3 = 0, \quad \frac{x_4^2}{a_4 - a_1} + \frac{x_5^2}{a_5 - a_1} + \frac{x_6}{a_6 - a_1} = 0.$$

Hence the surface belongs to the series $x_1 = x_2 = x_3 = 0$, but the spheres of this series also belong to the series $x_4 = x_5 = x_6 = 0$ and the spheres of the latter series are double, therefore, the surface of singularities consists of a Dupin cyclide counted twice.

If $x_1 = 0$ is the complex of points, the surface of singularities is a circle counted twice. By (A) this becomes a Dupin cyclide counted twice.

The Laguerre surface is a cone of revolution counted twice.

The complex [(111) 111] belongs to a series of five, the others being [(111)(11) 1], [(111) 12], [(111)(12)], [(111) 3], which are formed by aid of an involution [2] between the spheres on a Dupin cyclide. We find the four special cases.

1. Two double elements coincide.
2. Three double elements coincide.
3. Four double elements coincide.
4. Two pairs of double elements coincide.

We shall now show that the complex [(111) 111] consists of spheres which touch the corresponding spheres of a Dupin cyclide in an involution [2].

The complex may be written

$$a_4 x_4^2 + a_5 x_5^2 + a_6 x_6^2 = 0.$$

Any sphere of this complex is, therefore, given by

$$\sqrt{a_4}\, x_4 : \sqrt{a_5}\, x_5 : \sqrt{a_6}\, x_6 = \mu^2 - 1 : i(\mu^2 + 1) : 2\mu,$$

which divide the complex into a singly infinite number of linear congruences. Also, any sphere of the cyclide $x_1 = x_2 = x_3 = 0$ is given by

$$x_4 : x_5 : x_6 = \rho^2 - 1 : i(\rho^2 + 1) : 2\rho.$$

All spheres of such a linear congruence touch spheres of the cyclide for values of ρ given by

$$\frac{(\mu^2 - 1)(\rho^2 - 1)}{\sqrt{a_4}} - \frac{(\mu^2 + 1)(\rho^2 + 1)}{\sqrt{a_5}} + \frac{4\mu\rho}{\sqrt{a_6}} = 0.$$

If ρ_1 and ρ_2 are the roots of this quadratic we have

$$\rho_1 + \rho_2 = \frac{A\mu}{C\mu^2 + D}, \quad \rho_1\rho_2 = \frac{D\mu^2 + C}{C\mu^2 + D},$$

where

$$A = \frac{4}{\sqrt{a_6}}, \quad C = \frac{1}{\sqrt{a_5}} - \frac{1}{\sqrt{a_4}}, \quad D = \frac{1}{\sqrt{a_5}} + \frac{1}{\sqrt{a_4}}.$$

Hence, between ρ_1 and ρ_2, the following equation exists:

$$(\rho_1 + \rho_2)^2 = \frac{A^2}{(C^2 - D^2)^2}(C - D\rho_1\rho_2)(C\rho_1\rho_2 - D),$$

which defines the general involution [2].

The complex is, therefore, obtained by establishing a general involution between the spheres of a Dupin cyclide and taking all the spheres which touch each pair of corresponding spheres.

In the case [(111)(11) 1] $a_4 = a_5$ and, therefore, $c = 0$ and the involution becomes

$$(\rho_1 + \rho_2)^2 = \frac{A^2}{D^2}\rho_1\rho_2,$$

and, therefore, has two pairs of coincident roots, two zero and two infinite. This is a cubic involution.

6. [(111)(11) 1]. $\quad a_1 = a_2 = a_3 : a_4 = a_5.$

The surface of singularities in this case belongs to

$$(x_4 + ix_5)(x_4 - ix_5) = x_6 = 0.$$

It is a Dupin cyclide counted twice. The singular spheres of the complex are the spheres common to

$$a_4(x_4^2 + x_5^2) + a_6x_6^2 = 0, \; a_4^2(x_4^2 + x_5^2) + a_6^2x_6^2 = 0,$$

hence they form the congruences

$$x_4 + ix_5 = x_6 = 0 : x_4 - ix_5 = x_6 = 0.$$

The double spheres consist of one generation of the Dupin cyclide together with the two spheres common to

$$x_1 = x_2 = x_3 = x_6 = 0.$$

7. $$[(111)(111)] = a_1 = a_2 = a_3 : a_4 = a_5 = a_6 .$$

Since $\Pi\left(\frac{\partial F}{\partial x_i}\right) = 0$ for all values of x, which satisfy $F = 0$, the complex consists of the tangent pencils of spheres tangent to a Dupin cyclide.*

II.

Second Canonical Form. [11112]

1. $$\Pi = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5x_6 = 0 ,$$
$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + 2a_5x_5x_6 + x_5^2 = 0 ,$$
$$f = \frac{x_1^2}{k+a_1} + \frac{x_2^2}{k+a_2} + \frac{x_3^2}{k+a_3} + \frac{x_4^2}{k+a_4} + \frac{2x_5x_6}{k+a_5} - \frac{x_5^2}{(k+a_5)^2} = 0 .$$

The surface of singularities is the transform of the nodal cyclide, it is of order and class 20 and passes 10 times through K. It can be generated in five ways as the envelope of ∞^2 spheres, one of which is special, i. e. the linear complex of the congruence is a special complex. For this generation, the developable of bitangent planes becomes extraordinary (simple tangent planes). The transform of the conical point of the cyclide is a double sphere of the transformed surface.

If $x_5 = 0$ is the complex of planes, the surface is the Laguerre surface discussed by Smith.† That it is the Laguerre transform of the central quadric can be seen by making

$$x_1, x_2, x_3, x_4, x_5, x_6 = \xi, \eta, \zeta, \lambda, \mu_1, \nu_1 .$$

Then put $k = -a_4$ and we have

$$\lambda = 0, \frac{\xi^2}{a_1 - a_4} + \frac{\eta^2}{a_2 - a_4} + \frac{\zeta^2}{a_3 - a_4} + \frac{2\mu_1\nu_1}{a_5 - a_4} - \frac{\nu_1^2}{(a_5 - a_4)^2} = 0 .$$

Eliminating $\mu_1\nu_1$ by means of Π and dividing by ν, we have the point equation of a central quadric. By an (E), this quadric transforms into the surface discussed by Smith in the Annals.

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* Snyder, Bulletin of the American Mathematical Soc., Vol. 4, page 146.

† Annals of Mathematics, ser. 2, Vol. 1, page 153. This surface was also discussed by E. Müller, Monatshefte der Math. u. Physik, 1898, Vol. 9, "Die Geometrie orientierter Kugeln nach Grassmannschen Methoden," cf. 7, page 294 ff.

2. [(11) 112]. $a_1 = a_2$.

In 1, if $x_1 = 0$ is the complex of points, the surface is a nodal cyclide, and the focal line on the fundamental sphere of x_2 has a double point. [(11) 112] is then this focal line which, by (A), transforms into an annular surface with two double directrices and a double generator. It is of order and class 12, has two general developables of bitangent planes, one extraordinary one and finally one which degenerates into four planes. The locus of centers is a quartic with a double point. The surface can be generated in three ways as the envelope of ∞^2 spheres, one of which is special, and in one way as the envelope of ∞^1 spheres. This surface is also the transform of the central conic since 11112 is a quadric if $x_1 = \lambda$, $x_5 = \nu_1$.

If the focal line lies on the sphere whose points transform into planes, the surface becomes the developable of order 6, circumscribed to a sphere along a nodal sphero-quartic. This is the Laguerre transform of a cone with a minimum tangent plane. The surface has two double conics, its lines of curvature are of order 12.

If $x_5 = 0$ is the complex of planes, the surface is the Laguerre transform of the central quadric of revolution. It is an annular surface of order 8, class 4, has two plane directrices and, consequently, the lines of curvature are plane curves of order 8. It has two finite conics and K for double lines. If $x_1 = 0$ is the complex of points, then the surface reduces to a conic. By an (E) the latter becomes the same surface as the transform of the quadric of revolution.

It is to be noted here that there are two entirely distinct Laguerre surfaces corresponding to the same symbol [(11) 112]. If the complex of planes is one of the special complexes belonging to the congruence $x_1 = 0$, $x_2 = 0$, the surface is enveloped by ∞^1 planes. By no Laguerre transformation can one be transformed into the other.

3. [111 (12)]. $a_4 = a_5$.

The general nodal cyclide has a focal line on the point sphere x_5. If x_4 be the complex of points, the cyclide reduces to this focal line. In this case, c_4 cuts each generator of the cone in two points, and the spheres on which c_4 lies coincide in the point sphere x_5. By (A), c_4 transforms into a surface with coincident double directrices. It is of order and class 16, passes 8 times through K. It can be generated in three ways as the envelope of ∞^2 spheres, it has two general

developables of bitangent planes and one which degrades. The double curves are three sphero-quartics and the circle of contact with the directrix which is tacnodal.

Since the focal line lies on a point sphere, the developable cannot be generated as in the preceding cases, for (I) would be the only transformation that could be used and this would transform c_4 into a minimum developable. By an (I) with center at x_5, the cyclide [111 (12)] transforms into a quadric cylinder which by an (E) transforms into a developable of order 8.

In this case the directrices coincide in the plane at infinity. The developable has three finite double conics, and since parallel tangent planes of a cylinder transform into parallel planes, through a line at infinity, which is tangent to the surface, pass two planes of the developable and, consequently, the curve at infinity must be double; as the total double curve is of order 8, that in the plane at infinity must be a conic.

If $x_4 = 0$ is the complex of points and $x_5 = 0$ the complex of planes, the Laguerre surface [11112] is a general quadric. Consider the quadric as generated by its tangent planes, then [111 (12)] will be the point planes which circumscribe the quadric, i. e. the minimum planes which are tangent to the quadric. Hence, the surface [111 (12)], when $x_4 = 0$ is complex of points, is the focal developable of the general quadric. It is of order 8, class 4, has three finite conics and K for double lines. This is the minimum developable into which (I) transforms the focal line on the point sphere x_5.

4. [(11)(11) 2]. $\qquad a_1 = a_2, \quad a_3 = a_4.$

The cyclide [(11) 112) has a focal line on x_4 which breaks up into a circle and two minimum lines. Therefore, the surface of singularities of the general complex [(11)(11) 2] consists of a Dupin cyclide and a skew quadrilateral of minimum lines.

The developable Laguerre surface consists of a cone of revolution and two finite minimum lines and the infinite line in their plane counted twice.

When $x_5 = 0$ is the complex of planes, we obtain immediately from the equations that the Laguerre surface consists of a cone of revolution and a quadrilateral inscribed in K.

5. [1 (11)(12)]. $a_2 = a_3, \quad a_4 = a_5.$

The cyclide [111 (12)] has two tangent circles for focal line on the sphere x_3, therefore, the surface of singularities of the complex [1 (11)(12)] consists of two Dupin cyclides which have two consecutive spheres and, consequently, a circle in common.

When the complex of planes belongs to the congruence $x_2 = 0$, $x_3 = 0$, the Laguerre surface consists of two cones of revolution, tangent along a generator.

When $x_5 = 0$ is the complex of planes, the surface consists of two cylinders of revolution.

If $x_4 = 0$ is complex of points and $x_5 = 0$ complex of planes, the surface becomes the focal developable of the general quadric of revolution. It consists of two minimum cones having their vertices at the foci of a meridian section.

6. [(111) 12]. $a_1 = a_2 = a_3.$

The spheres of the series $x_4 = x_5 = x_6 = 0$ are double for the surface, therefore, the surface of singularities is a Dupin cyclide counted twice.

The Laguerre surface is a cone of revolution counted twice.

The complex may be written

$$a_4 x_4^2 + 2a_5 x_5 x_6 + x_5^2 = 0.$$

It is seen that any sphere of the complex is given by

$$x_4 : x_5 : x_6 = \sqrt{a_5}\,(2\mu a_5 - i) : \sqrt{a_4 a_5} : -\,2\mu\sqrt{a_4}\,(\mu\sqrt{a_5} - i),$$

giving ∞^1 linear congruences; the sphere $(0, 0, 0, 2\rho, 2, \rho^2)$ is any sphere of the cyclide $x_1 = x_2 = x_3 = 0$ and is touched by the spheres of the preceding congruence, provided that

$$\sqrt{a_4 a_5}\,\rho^2 + 4\sqrt{a_4}\,\mu\,(\mu\sqrt{a_5} - i) - 2\rho\sqrt{a_5}\,(2\mu\sqrt{a_5} - i) = 0.$$

This equation defines an involution [2], which has two coincident double elements.

7. [(111)(12)]. $a_1 = a_2 = a_3, \; a_4 = a_5.$

In this case the involution is defined by

$$(\rho_1 - \rho_2)^2 + \frac{a_4}{4} = 0,$$

which has all its double elements coincident.

8. [11 (112)]. $a_3 = a_4 = a_5,$

$$F = a_1 x_1^2 + a_2 x_2^2 + x_5^2 = 0.$$

The singular spheres are those which belong to $F = 0$ and satisfy

$$a_1^2 x_1^2 + a_2^2 x_2^2 = 0.$$

Thus the singular surface consists of the Dupin cyclide $x_1 = x_2 = x_5 = 0$, which degrades into a skew quadrilateral of minimum lines. The Laguerre surface consists of a quadrilateral inscribed in K. The complex is composed of the ∞^1 congruences

$$x_1 + i x_2 = 2\rho x_5,\ x_1 - i x_2 = 2\sigma x_5,$$

where ρ and σ are connected by the relation

$$a_1 (\rho + \sigma)^2 - a_1 (\rho - \sigma)^2 + 1 = 0.$$

Hence the complex is composed of spheres which touch corresponding spheres of two pencils which are in (2, 2) correspondence. The correspondence has two of its four double elements in coincidence

9. [(11)(112)]. $a_1 = a_2, \quad a_3 = a_4 = a_5.$

This complex consists of the singly infinite number of congruences

$$2\sqrt{a_1}\,(x_1 + i x_2) = \mu x_5, \quad 2\sqrt{a_1}\,(x_1 - i x_2) = -\frac{1}{\mu}\,x_5.$$

The directrices of these congruences form a tangent pencil of spheres and are in (1, 1) correspondence. Therefore, the complex consists of those spheres which touch corresponding spheres of two projective tangent pencils. The surface of singularities is same as [11 (112)].

III.

Third Canonical Form.

1. [1113]

$$F = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 (x_5^2 + 2x_4 x_6) + 2x_4 x_5 = 0,$$

$$\Pi = x_1^2 + x_2^2 + x_3^2 + x_5^2 + 2x_4 x_6 = 0,$$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{x_3^2}{k + a_3} + \frac{x_5^2 + 2x_4 x_6}{k + a_4} - \frac{2x_4 x_5}{(k + a_4)^2} + \frac{x_4}{(k + a_4)^3} = 0.$$

The surface [1113] is the transform of the cyclide with a general biplanar point. The biplanar point transforms into a cuspidal sphere, as may be seen by considering the transform of every plane section through this point. They will all have the transform of this point for cuspidal sphere and will envelope the surface. It is of order and class 18, can be generated in four ways as the envelope of a doubly infinite number of spheres, one of which is special. The surface is the envelope of those spheres of the complex which touch a fixed sphere of the complex; it has three general developables of bitangent planes and one extraordinary.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the paraboloid, as can be seen by transforming one of the fundamental complexes $x_1 = 0$, say, into the complex of points. Then the surface becomes the envelope of

$$\frac{x_2^2}{a-a} + \frac{x_3^2}{a-a} + \frac{2x_4x_5}{(a-a)^2} + \frac{x_4^2}{(a-a)^3} = 0, \quad x_1 = 0.$$

Or, using ξ, η, ζ, λ, μ, ν, the points of the complex envelope, the paraboloid

$$\frac{x^2}{a_2 - a} + \frac{y^2}{a_3 - a_1} - \frac{2z}{(a_4 - a_1)^2} + \frac{1}{(a_4 - a_1)^3} = 0.$$

Since the paraboloid has two planes of symmetry, the surface is sibireciprocal under three inversions and, therefore, has three double conics. Since the paraboloid is tangent to the plane at infinity, the transformed surface will have K for simple line. It is of order 10, class 4.

2. [(11) 13]. $a_1 = a_2$.

The cyclide [1113] has for focal line on the fundamental sphere of $x_2 - 0$, a quartic c_4 with a cusp, then the general surface [(11) 13] will be an annular surface which has two double spherical directrices and a cuspidal generator. It is of order and class 10, can be generated in two ways as the envelope of ∞^2 spheres, one of which is special and in one way as the envelope of ∞^1 spheres, it has one general developable of bitangent planes, one extraordinary, and one which degrades. The locus of centers is a skew quartic with a cusp.

If the quartic lies on the sphere whose points transform into planes, the surface becomes the developable of order 5, touching a sphere along a sphero-

quartic with a cusp. It has one double conic. The cuspidal edge is a quartic having a cusp.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the paraboloid of revolution or of the parabola. It is of order 4, class 5, contains one double conic.

3. [11 (13)]. $a_3 = a_4$.

The cyclide [1113] has a focal line on the point sphere x_4 (minimum cone having x_4 for vertex), which has a double point at the vertex. Each generator of the cone cuts the quartic in but one point aside from the vertex. Then the general surface [11 (13)] will be an annular surface with a single directrix, which is also a double generator. It is of order and class 12.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the parabolic cylinder. It is of order 6, class 4, has two doube conics. The director sphere coincides with the plane at infinity.

If $x_4 = 0$ is the complex of planes and $x_3 = 0$ complex of points, the surface is the focal developable of the paraboloid. It is of order 6, class 4, has two finite double conics; K is simple line on the surface.

4. [1 (113)].

$$a_2 = a_3 = a_4\,,$$
$$F = a_1 x_1^2 + 2x_4 x_5 = 0\,.$$

The singular spheres satisfy $F = 0$ and

$$a_1^2 x_1^2 + x_4^2 = 0\,.$$

Thus the surface of singularities consists of the Dupin cyclide $x_1 = x_4 = x_5 = 0$, which degrades into a skew quadrilateral of minimum lines.

The spheres of the complex belong to the single infinite number of linear congruences,

$$a_1(x_5 + ix_1) + 2\mu x_4(a_1\mu - i) = 0\,,$$
$$a_1(x_5 - ix_1) + 2\mu x_4(a_1\mu + i) = 0\,.$$

The directrices of these congruences form two pencils in (2, 2) correspondence. Thus the complex is made up of spheres which touch corresponding spheres of two tangent pencils in (2, 2) correspondence.

5. $[(11)(13)]$. $a_1 = a_2, \quad a_3 = a_4.$

The focal line of the cyclide $[11\,(13)]$ on the sphere x_2 breaks up into a circle and two minimum lines which intersect on it, therefore, the general surface $[(11)(13)]$ consists of a Dupin cyclide and skew quadrilateral of minimum lines lying on one of its generating spheres.

The Laguerre surface consists of a cone of revolution and two minimum lines lying in one of its generating planes.

If $x_3 = 0$ is the complex of points, the surface is the focal developable of the paraboloid of revolution. It is a minimum cone with focus as vertex.

6. $[(111)\,3]$ $a_1 = a_2 = a_3$

The surface of singularities is a Dupin cyclide counted twice.

The singular spheres form the special congruences $x_4 = x_5 = 0$, together with a general linear congruence.

The spheres of the complex are given by the equations

$$x_4 : x_5 : x_6 = a_4^2 : \; 2\mu a_4 : -\; 2\mu\,(1 - \mu\surd a_4),$$

forming ∞^1 linear congruences; the spheres of such a congruence touch the sphere $(0, 0, 0, 2\rho^2, 2\rho, -1)$ of the cyclide $x_1 = x_2 = x_3 = 0$, provided that

$$4\mu\rho^2\,(1 + \mu\surd a_4) - 4\mu a_4\rho + a_4^2 = 0.$$

If the roots of this equation are ρ_1, ρ_2, it is seen that

$$(\rho_1 - \rho_2)^2 + \frac{4}{a_4}\,\rho_1\rho_2\,(\rho_1 + \rho_2) = 0$$

this is an involution [2], in which three double elements coincide. The Laguerre surface is a cone of revolution counted twice.

IV.

Fourth Canonical Form. [1122]

1. $F = a_1x_1^2 + a_2x_2^2 + 2a_3x_3x_4 + 2a_4x_5x_6 + x_3^2 + x_5^2 = 0,$

$\Pi = x_1^2 + x_2^2 + 2x_3x_4 + 2x_5x_6 = 0,$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{2x_3x_4}{k + a_3} - \frac{x_3^2}{(k + a_3)^2} + \frac{2x_5x_6}{k + a_4} - \frac{x_5^2}{(k + a_4)^2} = 0.$$

The surface of singularities is the transform of the non-annular binodal cyclide, therefore, it has two double spheres which touch each other. It is of order and class 16, can be generated in four ways as the envelope of ∞^2 spheres, two of which are special; it has two general developables of bitangent planes and two extraordinary.

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface reduces to a quadric, which is tangent to K at one point, then by a Laguerre transformation this quadric inverts into a surface having a double plane. It is of order 10, class 4, has K for double line and has two finite double conics.

2. [(12) 12]. $a_1 = a_2$.

The cyclide [1212] has on the point sphere x_3 a focal line which has a double point, not at the vertex of the minimum cone. Therefore, the general surface is an annular surface which has coincident directrices and a double generator. It is of order and class 12, can be generated in two ways as the envelope of ∞^2 spheres, one of which is special; there are two developables of bitangent planes, one of which is extraordinary. The developable corresponding to the generation as the envelope of ∞^1 spheres degrades into planes.

If $x_3 = 0$ is the complex of planes and $x_2 = 0$ the complex of points, the surface is a quadric cylinder which is tangent to K in one point. By a Laguerre transformation this inverts into a developable of order 6, class 4, has one finite double conic and one in the plane at infinity.

If $x_1 = 0$ is complex of points the surface is the focal developable of a quadric which touches K. It is of order 6, class 4, has one finite conic and K for double curve.

If $x_5 = 0$ is the complex of planes and $x_2 = 0$ the complex of points, the surface becomes a quadric of revolution such that the two points of contact with K coincide. By a Laguerre transformation this inverts into an annular surface with coincident plane directrices, it is of order 8, class 4, contains K as double line and has one finite double conic.

3. [(12)(12)]. $a_1 = a_2,\ a_3 = a_4$.

The cyclide [12 (12)] has a focal line on the point sphere x_3 consisting of two tangent circles. Therefore, the surface of singularities of the general complex [(12)(12)] consists of two Dupin cyclides which touch along a circle (since

they have consecutive generating spheres in common), and as the two directrices coincide, they must intersect along a circle of the second system, i. e. the cyclides have a circle of each system in common.

If $x_3 = 0$ is the complex of planes, the surface is given by

$$\frac{x_1^2 + 2x_3x_4}{a_1} + \frac{x_3^2}{a_1^2} + x_6^2 = 0 \quad x_3 = 0,\, x_5 = 0,$$

which is two lines lying in a minimum plane. By a Laguerre transformation, these transform into two cones of revolution having a generating sphere and *one* generating plane in common.

4. $[(11)\,22]$. $a_1 = a_2$.

The cyclide $[1122]$ has a focal line on the sphere x_2 which degrades into a cubic and a double secant. Therefore, the general surface has a single and a double directrix. It consists of an annular surface of order and class 8, and two minimum lines, one on each directrix, which belong to two non-consecutive generators of the annular surface.

If the sphere x_2 is the sphere whose points transform into planes, the cubic inverts into a developable of order 4, class 3, and the line inverts into a minimum line which lies on the directrix sphere of the developable.

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface becomes a conic passing through one of the circle points. By a Laguerre transformation this becomes an annular surface of order 6, class 4.

5. $[11\,(22)]$. $a_3 = a_4$.

The cyclide $[11\,(22)]$ is ruled and has a double minimum line. By a transformation (A), the minimum line d transforms into two lines d', d'', and a line cutting d transforms into two lines, one cutting d' and one cutting d'', and the pair of lines, cutting d in the same point, transform into two pairs of lines, one pair intersecting on each line d' and d''. Two lines intersecting on K invert into two pairs of lines, each intersecting on K and one line of the pair cuts d' and the other d'', therefore, there is a $(2, 2)$ correspondence between d' and K and between d'' and K. Hence, the surface of the singularities consist of two ruled cyclides.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface is a cone with vertex on K, and by an (E), it transforms into two such cones.

The equation of the complex is

$$a_1 x_1^2 + a_2 x_2^2 + x_3^2 + x_5^2 = 0,$$

and, therefore, can be broken up into the congruences

$$\begin{aligned} -\mu\sqrt{a_1 a_2}\,x_1 - i\mu\sqrt{a_1 a_2}\,x_2 &+ (\mu^2\sqrt{a_1}(\sqrt{a_1}+\sqrt{a_2}) + \sqrt{a_1} - \sqrt{a_2})\,x_3 \\ &+ i(\mu^2\sqrt{a_1}\sqrt{a_1}+\sqrt{a_2}) - \sqrt{a_1} + \sqrt{a_2})\,x_3 = 0, \\ \mu\sqrt{a_1 a_2}\,x_1 + i\mu\sqrt{a_1 a_2}\,x_2 &+ (\mu^2\sqrt{a_1}(\sqrt{a_1}-\sqrt{a_2}) + \sqrt{a_1} + \sqrt{a_2})\,x_3 \\ &+ i(\mu^2\sqrt{a_1}(\sqrt{a_1}-\sqrt{a_2}) - \sqrt{a_1} - \sqrt{a_3})\,x_5 = 0. \end{aligned}$$

The directrices of these congruences form two ruled cyclides which are projective with each other. Therefore, the complex consists of those spheres which touch corresponding spheres of two ruled cyclides which are projective with each other.

6. [(112) 2]. $\qquad a_1 = a_2 = a_3.$

The surface of singularities belongs to the series $x_3 = x_4 = x_5 = 0$, and, therefore, consists of a skew quadrilateral of minimum lines counted twice.

If $x_5 = 0$ is complex of planes, the Laguerre surface is a quadrilateral inscribed in K. If $x_3 = 0$ is complex of planes, the Laguerre surface is two finite minimum lines which intersect and the line at infinity in their plane counted twice.

The complex is formed of spheres which belong to the congruence

$$\mu x_3 + x_5 = 0, \quad 2\mu a_3 x_4 - (\mu^2 + 1)\,x_5 = 0,$$

and is, therefore, formed of spheres which touch paired spheres of two pencils in (1, 2) correspondence, which have a common self-corresponding element.

7. [1 (122)]. $\qquad a_2 = a_3 = a_4.$

The singular spheres belong to the series $x_1 = x_3 = x_5 = 0$, and the surface is two minimum lines, each counted four times.

If $x_3 = 0$ is the complex of planes, the surface becomes a finite minimum line and a line in the plane at infinity cutting it, each counted four times.

8. [(11)(22)]. $\qquad a_1 = a_2,\ a_3 = a_4.$

In the cyclide [11 (22)] there is a focal line on the sphere x_3 consisting of two skew lines and one of their secants counted twice. Therefore, the surface of singularities of the general complex [(11)(22)] consists of two skew quadrilaterals of minimum lines having two common sides. The complex is formed of spheres which belong to the complexes

$$a\mu\,(x_1 + ix_2) + x_3 + ix_5 = 0\,,$$
$$a_1\,(x_1 - ix_2) - \mu\,(x_3 - ix_5) = 0\,.$$

The directrices of these congruences form two projective tangent pencils of spheres so related that one of the minimum lines enveloped by the first pencil cuts one of the minimum lines enveloped by the second pencil.

V.

Fifth Canonical Form. [114]

1. $F_1 = a_1x_1^2 + a_2x_2^2 + 2a_3\,(x_3x_6 + x_4x_5) + 2x_3x_5 + x_4^2 = 0\,,$

$$\Pi = x_1^2 + x_2^2 + 2x_3x_6 + 2x_4x_5 = 0\,,$$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{2x_3x_6 + 2x_4x_5}{k + a_3} - \frac{2x_3x_5 + x_4^2}{(k + a_3)^2} + \frac{2x_3x_4}{(k + a_3)^3} - \frac{x_3^2}{(k + a_3)^4} = 0\,.$$

The surface of singularities is the transform of the cyclide with a special biplanar point. It is, therefore, of order and class 16. The biplanar point inverts into a sphere analogous to a tacnode, as we shall see in the case [(11) 4]. The surface can be generated in three ways as the envelope of ∞^2 spheres, one of which is special; it has two general developables of bitangent planes and one extraordinary. The surface is the envelope of spheres which touch a singular sphere of a quadratic complex (obtained by putting $k = -a_3$).

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface reduces to a paraboloid, which is tangent to the plane at infinity at a point of K. By a Laguerre transformation, this inverts into a surface of order 9, class 4, containing K as simple line; it has two double conics.

2. [(11) 4]. $a_1 = a_4$.

The cyclide [114] has a focal line on the sphere x_3 consisting of a twisted cubic and a minimum line tangent to it. Therefore, the general surface consists of an annular surface of order 8, and two minimum lines lying on two consecutive generating spheres. The surface can be generated in one way as the envelope of ∞^2 spheres; it has no general developable of bitangent planes, but has one extraordinary and one which degrades.

If the cubic and tangent lie on the sphere whose points transform into planes, the surface becomes a developable of order 4 and a minimum line.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ is the complex of points, the surface becomes a parabola tangent to the line at infinity in one of the circle points. By a Laguerre transformation this inverts into a surface of order 5, containing K as simple line.

3. [1 (14)]. $a_2 = a_3$.

The cyclide [114] has on the point sphere x_3 a focal line having a cusp at the vertex of the minimum cone. Therefore, the general surface [1 (14)] is an annular surface with a single directrix which is also a cuspidal generator, it is of order and class 10.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface becomes a parabolic cylinder whose generator in the plane at infinity is tangent to K. By a Laguerre transformation this becomes a developable of order 5, class 4, having one double conic. The cuspidal edge is a quartic having a cusp.

If $x_2 = 0$ is the complex of points, the surface is the focal developable of a paraboloid which is tangent to the plane at infinity in a point of K. It is of order 5, class 4, has one double conic and contains K as simple line.

4. [(114)]. $a_1 = a_2 = a_3$.

The surface of singularities belongs to $x_3 = x_4 = x_5 = 0$ and, therefore, consists of a skew quadrilateral of minimum lines counted twice.

If $x_3 = 0$ is the complex of planes, the surface is a quadrilateral inscribed in K.

The complex consists of the singly infinite number of congruences

$$2x_3 - \mu x_4 = 0, \quad 2x_3 - \mu^2 x_5 = 0.$$

The directrices of these special congruences form two tangent pencils in (2, 1) correspondence having a self-corresponding element. The involution formed by the pair of elements of the first pencil corresponding to the elements of the second pencil, has the common self-corresponding element for double element. The complex is, therefore, composed of those spheres which touch corresponding spheres of two tangent pencils in (2, 1) correspondence having a self-corresponding element.

VI.

Sixth Canonical Form. [123]

1. $$F = a_1x_1^2 + 2a_2x_2x_3 + x_2^2 + a_3(2x_4x_6 + x_5^2) + 2x_4x_5 = 0,$$
$$\Pi = x_1^2 + 2x_2x_3 + 2x_4x_6 + x_5^2 = 0,$$
$$f = \frac{x_1^2}{k+a_1} + \frac{2x_2x_3}{k+a_2} - \frac{x_2^2}{(k+a_2)^2} + \frac{2x_4x_6 + x_5^2}{k+a_3} - \frac{2x_4x_5}{(k+a_3)^2} + \frac{x_4^2}{(k+a_3)^3} = 0.$$

The surface is the transform of the cyclide, which has a conical and a general biplanar point; it, therefore, has a double and cuspidal sphere which touch. It is of order and class 14, can be generated in three ways as the envelope of ∞^2 spheres, two of which are special, and has one general developable of bitangent planes and two extraordinary ones.

If $x_2 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface becomes a quadric which has three points contact with K. By a Laguerre transformation, this becomes a surface of order 9, class 4, has a cuspidal plane, has K and a finite conic for double curves.

If $x_4 = 0$ is the complex of planes the surface becomes a paraboloid tangent to K. By a Laguerre transformation, this becomes a surface of order 8, class 4, containing K as simple line. It has one double conic and one double plane.

2. [(12) 3]. $a_1 = a_2$.

The cyclide [123] has a focal line on the point sphere x_2 which has a cusp not at the vertex of the minimum cone. Therefore, the general surface [(12) 3] is an annular surface which has coincident directrices and a cuspidal generator. It is of order and class 10.

If $x_4 = 0$ is the complex of planes, and $x_1 = 0$ the complex of points, the surface becomes a general parabola lying in a minimum plane; by a Laguerre transformation this becomes an annular surface with coincident plane directrices. It is of order 6, class 4, and contains K as simple line.

If x_2 is on the fundamental sphere of (A) the surface becomes one with a five-fold point at x_2. By a transformation by reciprocal radii, this inverts into a developable of order 5 (the Laguerre surface if $x_2 = 0$ is complex of planes).

3. [(13) 2]. $\qquad a_1 = a_3.$

The cyclide [123] has a focal line on x_4 consisting of a cubic and a secant. The general surface is the same as [(11) 22] except that the directrices coincide.

If $x_4 = 0$ is the complex of planes, the developable is of order 4.

If $x_1 = 0$ is the complex of points, the developable becomes the focal developable of a paraboloid tangent to K. It is of order 4, class 3, has K for simple line.

If $x_2 = 0$ is the complex of planes, the annular surface is of order 6. For if x_2 is on the fundamental sphere of (A), the transformed surface will have in x_2 a four-fold point arising from the union of two conical points, but the minimum line goes through this point, therefore, the annular surface has this point for conical point, and by an (I) with centre at x_2, the surface inverts into an annular surface of order 6.

4. [1 (23)]. $\qquad a_2 = a_3.$

The cyclide [1 (23)] is ruled, and for same reason as [11 (22)], the general surface consists of two such cyclides. The double line in this cyclide is simple directrix and simple generator.

The Laguerre surface, as in [11 (22)], consists of two parabolic cylinders with vertex on K.

5. [(123)].

$$a_1 = a_2 = a_3,$$
$$F = x_2 - 2x_4x_5 = 0,$$
$$\Pi\left(\frac{\partial F}{\partial x_i}\right) = x_4^2 = 0.$$

The surface of singularities belongs to the series $x_2 = x_4 = x_5 = 0$ and since Δ*

* See Snyder, "Criteria for Nodes," etc., Annals of Math., 1897.

and its first minors vanish, the quadrilateral becomes two intersecting minimum lines counted four times.

If $x_2 = 0$ or $x_4 = 0$ is the complex of planes, the surface is a line at infinity counted four times.

VII.

Seventh Canonical Form. [15]

1. $\Pi = x_1^2 + 2x_2x_6 + 2x_3x_5 + x_4^2 = 0,$

$$F = a_1x_1^2 + a_2(2x_2x_6 + 2x_3x_5 + x_4^2) + 2x_2x_5 + 2x_3x_4 = 0,$$

$$f = \frac{x_1^2}{k+a_1} + \frac{2x_2x_6 + 2x_3x_5 + x_4^2}{k+a_2} - \frac{2x_2x_5 + 2x_3x_4}{(k+a_2)^2} + \frac{2x_2x_4 + x_3^2}{(k+a_2)^3} - \frac{2x_2x_3}{(k+a_2)^4} + \frac{x_2^2}{(k+a_2)^5} = 0.$$

The surface of singularities is the inverse of the cyclide, which has a special biplanar point. The surface is of order and class 14; it can be generated in two ways as the envelope of ∞^2 spheres, one of which is special, therefore, it has one general developable of bitangent planes and one extraordinary one.

If $x_1 = 0$ is complex of points and $x_2 = 0$ the complex of planes, the surface becomes a paraboloid, which is tangent to the plane at infinity in a point of K, and one of the lines in the plane at infinity is tangent to K. By a Laguerre transformation, this transforms into a surface of order 8, class 4, having one double conic and having K for simple line.

2. [(15)]. $\qquad a_1 = a_2.$

The cyclide [15] has a focal line on the point sphere x_2 consisting of a twisted cubic and a minimum line tangent to it. The general surface consists of an annular surface as in [(11) 4] except that the directrices coincide.

The Laguerre surface is a developable of order 4, and a minimum line. This differs from [(21) 4] in having the director sphere coincide with the plane at infinity.

If $x_1 = 0$ is the complex of points and $x_2 = 0$ the complex of planes, the Laguerre surface becomes the focal developable circumscribed to the paraboloid [15]. It is of order 4, contains K as simple line.

VIII

Eighth Canonical Form. [222]

1. $F = 2a_1x_1x_2 + 2a_2x_3x_4 + 2a_3x_5x_6 + x_1^2 + x_3^2 + x_5^2 = 0,$
$\Pi = x_1x_2 + x_3x_4 + x_5x_6 = 0.$

If $x_5 = 0$ is the complex of planes the surface is the Laguerre surface having two double planes, and as the Laguerre surfaces are of class 4, two double planes will reduce the order of the surface to 8. The minimum line common to $x_1 = x_3 = x_5 = 0$ is also a part of the surface, since these spheres are all singular spheres. The presence of this line does not reduce the order nor class of the residual surface.

By an (I), this inverts into the general surface of singularities, which is, therefore, of order and class 12, passes six times through K. The minimum line enveloped by $x_1 = x_3 = x_5 = 0$ is also a part of this surface. The surface can be generated in three ways as the envelope of ∞^2 spheres, two of which are special. It has three double spheres which have a minimum line in common.

2. [2 (22)]. $a_2 = a_3,$

$$F = 2a_1x_1x_2 + x_1^2 + x_3^2 + x_5^2 = 0,$$
$$f = \frac{2x_1x_2}{a_1 - a_2} - \frac{x_1^2}{(a_1 - a_2)^2} + x_4^2 + x_6^2 = 0, \; x_3 = 0, \; x_5 = 0.$$

Since $x_3 = 0$, $x_5 = 0$, we have at once from Π either $x_1 = 0$ or $x_2 = 0$. In the first case, spheres which belong to $x_1 = x_3 = x_5 = 0$ and f, from two tangent pencils of spheres which have a sphere in common. In the second case, spheres which belong to $x_2 = x_3 = x_5 = 0$ and f, form a ruled cyclide. Hence, the surface of singularities consists of two tangent pencils of spheres and a ruled cyclide.

The complex may be broken up into the congruences

$$x_1 - \mu x_3 - i\mu x_5 = 0,$$
$$2a_1\mu x_2 + (1 + \mu^2)x_3 - i(1 - \mu^2)x_5 = 0,$$

the directrices of these congruences are

$$S' \equiv (0, 1, 0, -\mu, 0, -i\mu),$$
$$S \equiv (2a\mu, 0, 0, 1 + \mu^2, 0, -i(1 - \mu^2)).$$

The spheres S' form a tangent pencil. The spheres S form a ruled cyclide. Therefore, the complex consists of spheres which touch paired spheres (S', S).

The pencils and the cyclide have a self-corresponding sphere. If the sign of i be changed, i. e. if the congruences be found in another way, we get the same ruled cyclide but a different pencil. The two pencils have a sphere in common, thus the complex can be generated in two ways as above.

3. [(222)].

$$a_1 = a_2 = a_3.$$
$$F = x_1^2 + x_3^2 + x_5^2 = 0,$$
$$\Pi\left(\frac{\partial F}{\partial x_i}\right) = 0.$$

Every sphere of the complex is singular, and as x_1, x_3, x_5 are special and in involution, the surface is generated by spheres which have a minimum line in common and which satisfy a quadratic complex. This is the ruled cyclide. For, if S_1, S_3, S_5 be three such spheres, then any sphere can be represented by

$$S_i = x_2 s_1 + x_4 s_3 + x_6 s_5 = 0,$$
$$\pi = x_1 x_2 + x_3 x_4 + x_5 x_6 = 0.$$

The spheres S_i belong to $x_1 = x_3 = x_5 = 0$ and, therefore, the envelope subject to the quadratic relation is

$$a_1' a_2' s_1^2 + a_1' s_3^2 + a_2' s_5^2 = 0,$$

which is a ruled cyclide, since s_1, s_3, s_5 have a minimum line in common.

IX.

Ninth Canonical Form. [33]

1. $$F = a_1(2x_1x_3 + x_2^2) + 2x_1x_2 + a_2(2x_4x_6 + x_5^2) + 2x_4x_5 = 0,$$
$$\Pi = 2x_1x_3 + x_2^2 + 2x_4x_6 + x_5^2 = 0,$$
$$f = \frac{2x_1x_3 + x_2^2}{k+a} - \frac{2x_1x_2}{(k+a_1)^2} + \frac{x_1^2}{(k+a_1)^3} + \frac{2x_4x_6 + x_5^2}{k+a_2} - \frac{2x_4x_5}{(k+a_2)^2} + \frac{x_4^2}{(k+a_2)^3} = 0.$$

If $x_4 = 0$ is the complex of planes, the surface is a Laguerre surface having a cuspidal plane. The surface [1113] is of order 10, and as a cuspidal plane reduces the order of the surface three, the order of the surface [33] is 7.

By an (I), this inverts into the general surface of singularities, therefore, the general surface of singularities is of order and class 12, passes six times through K. It has two cuspidal spheres which touch each other.

2. $[(33)]$.
$$a_1 = a_2,$$
$$F = x_1x_2 + x_4x_5 = 0.$$

The complex may be decomposed into the congruences

$$x_1 + x_4\, 2\mu x_2' = 0, \quad x_1 - x_4 - 2i\mu x_5' = 0, \quad [x_2 = x_2' + ix_5', \; x_5' = x_2' - ix_5'].$$

The directrices of the congruences are

$$S_1 \equiv (0,\, 0,\, 1,\, 0,\, 2\mu,\, 1),$$
$$S_2 \equiv (0,\, -2i\mu,\, 1,\, 0,\, 0,\, -1),$$

and form two tangent pencils of spheres in (1, 1) correspondence such that the sphere $T_1 \equiv (0, 0, 1, 0, 0, -1)$ touch all the spheres S_1, and the sphere $T \equiv (0, 0, 1, 0, 0, 1)$ touch all the spheres S_2.

If the congruences were

$$x_1 + \mu x_4 = 0, \quad \mu x_2 - x_5 = 0,$$

the directrices all coincide in the pencil. $S \equiv (0, 0, 1, 0, 0, \mu)$. The two minimum lines enveloped by this pencil are such one belongs to S_1 and the other to S_2. Therefore, in the surface of singularities, these lines count as triple line and the other two as simple lines.

X.

Tenth Canonical Form. [24]

1.
$$F = 2a_1x_1x_2 + x_1^2 + a_2(2x_3x_6 + 2x_4x_5) + 2x_3x_5 + x_4^2 = 0,$$
$$\Pi = x_1x_2 + x_3x_6 + x_4x_5 = 0.$$

In this case there are two Laguerre surfaces according as x_1 or x_3 is the complex of planes. The first is a surface with a tacnodal plane, and, therefore, of order 8, it has K for double line. The second differs from [114] by having a double plane and, therefore, is of order 7, and has K for simple line.

By an (I), these invert into the general surface which is, therefore, of order and class 12; has one double and one tacnodal sphere. In this case, as in the case [222], a minimum line forms part of the surface.

2. $[(24)]$.
$$a_1 = a_2,$$
$$F = x_1^2 + 2x_3x_5 + x_4^2 = 0,$$
$$f = x_3^2 + 2x_4x_6 + x_5^2 = 0, \quad x_1 = 0, \; x_3 = 0.$$

Either $x_4 = 0$ or $x_5 = 0$. In the first case spheres common to $x_1 = x_3 = x^4$ and f envelope two tangent pencils; in the second case, spheres common to $x_1 = x_3 = x_5$ and f form a ruled cyclide.

The complex consists of the congruences

$$2\mu x_3 - x_1 - ix_4 = 0, \quad 2\mu x_1 - 2\mu^2 x_3 + x_5 = 0.$$

The directrices of these special congruences are

$$S_1 \equiv (0, -1, 0, 0, -i, 2\mu),$$
$$S_2 \equiv (0, 2\mu, 0, 1, 0, -2\mu^2).$$

The spheres S_1 form a tangent pencil and the spheres S_2 form a ruled cyclide. S_1 has one sphere in common with S_2, viz. $(0, 0, 0, 0, 0, 2)$ and the sphere of S_1 touch no other sphere of S_2. The congruences might also have been

$$2\mu x_3 - x_1 + ix_4 = 0, \quad 2\mu x_1 - 2\mu^2 x_3 + x_5 = 0.$$

The directrices in this case are

$$S_1' \equiv (0, -1, 0, 0, i. 2\mu),$$
$$S_2 \equiv (0, 2\mu, 0, 1, 0, -2\mu^2).$$

The ruled cyclide is the same in each case, and S_1' and S_1 have the same sphere in common with the cyclide. The spheres S_1 touch all the spheres S_1', i.e. the two pencils have a minimum line in common.

The complex can be generated in two ways by spheres which touch corresponding spheres of a tangent pencil and a cyclide projectively related. The cyclide and the two tangent pencils form the surface of singularities.

XI.

Eleventh Canonical Form. [6]

1. $$\Pi = x_1x_6 + x_2x_5 + x_3x_4 = 0,$$
$$F = 2a_1(x_1x_6 + x_2x_5 + x_3x_4) + 2x_1x_5 + 2x_2x_4 + x_3^2 = 0,$$
$$f = \frac{2(x_1x_6 + x_2x_5 + x_3x_4)}{k + a_1} - \frac{2x_1x_5 + 2x_2x_4 + x_3^2}{(k + a_1)^2} + \frac{2x_1x_4 + 2x_2x_3}{(k + a_1)^3}$$
$$- \frac{2x_1x_3 + x_2^2}{(k + a_1)^4} + \frac{2x_1x_2}{(k + a_1)^5} - \frac{x_1^2}{(k + a_1)^6} = 0.$$

It was seen that the surface [1122] becomes the surface [114] when the two double spheres become consecutive, in a similar manner the surface [222]

becomes [24] when two of the double spheres become consecutive. Thus the surface [6] has three consecutive double spheres which have a minimum line in common; it is of order 12 and can be generated in one (special way as the envelope of ∞^2 spheres. The minimum line enveloped by $x_1 = x_2 = x_3 = 0$ is a part of the surface.

The Laguerre surface is of order 8, class 4.

XII.

The following table contains a complete list of those surfaces which appear as surfaces of singularities of the quadratic complex. The cyclide and spheroquartic are not included here, since they have been classified by Loria in the paper referred to.

The following symbols are used:

ν_1 = complex of planes.
Π = plane at infinity.
$(\bar{E})$ = Laguerre transform.
S_2 = central quadric.
$\bar{S}_2$ = paraboloid.
λ = complex of points.
quad. = skew quadrilateral of minimum lines.
C_4 = Dupin cyclide.
C_2 = cone of revolution.

NON-ANNULAR SURFACES.

General.	Order.	Class.	Laguerre.	Order.	Class.
111111. General surface,	24	24	No Laguerre surface.		
11112. One double sphere,	20	20	General Laguerre surface,	12	4
1113. One cuspidal sphere,	18	18	$(\bar{E})$ of $\bar{S}_2$,	10	4
114. One tacnodle sphere,	16	16	$(\bar{E})$ of $\bar{S}_2$, touching Π in point of K,	9	4
15. Singular sphere arises from union of double and cuspidal sphere,	14	14	$(\bar{E})$ of $\bar{S}_2$, touching Π in point of K; one generator in Π tangent to K,	8	4
1122. Two double spheres,	16	16	$(\bar{E})$ of S_2 touching K,	10	4
123. One double, one cuspidal sphere,	14	14	If $x_4 = \nu$, $(\bar{E})$ of $\bar{S}_2$, touching K,	8	4
			If $x_2 = \nu$, $(\bar{E})$ of S_2 having three point contact with K,	9	4
33. Two cuspidal spheres,	12	12	One cuspidal plane,	7	4

SURFACES WHICH CONTAIN A MINIMUM LINE AS PART OF THE ENVELOPE.

General.	Order.	Class.	Laguerre.	Order.	Class.
222. Three double spheres,	12	12	Two double planes,	8	4
24. One double, one tacnodal sphere	12	12	If $x_3 = \nu$, one double plane,	7	4
			If $x_1 = \nu$, one tacnodal plane,	8	4
6. Three coincident double spheres,	12	12		7	4

ANNULAR SURFACES.			DEVELOPABLES.		
(11) 1111. Two distinct double directrices,	16	16	$(\bar{E})$ of quadric cone,	8	4
111 (12). Coincident double directrices,	16	16	If $x_5 = \nu$, $(\bar{E})$ of cylinder,	8	4
			If $x_5 = \nu$, $x_4 = \lambda$, focal developable of central quadric,	8	4
11 (13). Single directrix which is also a double generator,	12	12	$(\bar{E})$ of parabolic cylinder,	6	4
			If $x_3 = \lambda$, focal developable of paraboloid,	6	4
1 (14). Single directrix which is cuspidal generator,	10	10	$(\bar{E})$ of parabolic cylinder in which line in Π touches K,	5	4
			If $x_2 = \lambda$ focal developable of $\bar{S}_2$ touching Π in point of K,	5	4
(11) 112. Two double directrices and double generator,	12	12	If $x_5 = \nu$, $(\bar{E})$ of $\bar{S}_2$ of revolution,*	8	4
			If $x_1 + ix_2 = \nu$, $(\bar{E})$ of cone with minimum plane,	6	4
(11) 13. Two double directrices and cuspidal generator,	10	10	If $x_4 = \nu$, $(\bar{E})$ of $\bar{S}_2$, of revolution,*	6	4
			If $x_1 + ix_2 = \nu$, $(\bar{E})$ of cone having three point contact with K,	5	4

* These surfaces are annular,

General.	Order.	Class.	Laguerre.	Order.	Class.
12 (12). Coincident double directrices and double generator,	12	12	If $x_5 = \nu$, $(\bar{E})$ of cylinder which touches K,	6	4
			If $x_3 = \nu$, $(\bar{E})$ of S_2 of revolution with minimum axis,*	8	4
			If $x_5 = \nu$, $x_2 = \lambda$, focal developable of S_2 which touches K,	6	4
(12) 3. Coincident double directrices and cuspidal generator,	10	10	If $x_3 = \nu$, $x_1 = \lambda$, focal developable of S_2 having three point contact with K,	5	4
			If $x_3 = \nu$, $(\bar{E})$ of above,*	5	4
			If $x_4 = \nu$, $(\bar{E})$ of parabola which lies in minimum plane,*	6	4

FACTORABLE ANNULAR AND DEVELOPABLE SURFACES.

(11) 22. One double, one single directrix,	8	8	If $x_5 = \nu$, $(\bar{E})$ of conic passing through one circle point,	6	4
			If $x_1 + ix_2 = \nu$, developable circumscribed to sphere along cubic,	4	3
(11) 4. One double, one single directrix,	8	8	If $x_3 = \nu$, $(\bar{E})$ of parabola passing through a circle point,	5	4
			If $x_1 = ix_2 = \nu$, developable circumscribed to a sphere along a cubic,	4	3
2 (13). Single directrix,	8	8	If $x_3 = \nu$, annular,	6	3
			If $x_4 = \nu$, developable,	4	3
			If $x_4 = \nu$, $x_1 = \lambda$, focal developable of S_2 tangent to K,	4	3

* These surfaces are annular.

General.	Order.	Class.	Laguerre.	Order.	Class.
(15). Single directrix,	8	8	If $x_2 = \nu$, $x_1 = \lambda$, focal developable of $\bar{S}_2$ touching Π in point of K and one generator in Π tangent to K,	4	3
			If $x_2 = \nu$, $(\bar{E})$ of above,	4	3
Two C_4.			*Two C_2.*		
(11)(11)11. Two C_4 having two spheres of each generation in common.			Two C_2 having two common planes.		
1(11)(12). Two C_4 touching along a circle.			If $x_2 + ix_3 = \nu$, two C_2 tangent along an element.		
			If $x_5 = \nu$, two cylinders.		
			If $x_5 = \nu$, $x_4 = \lambda$, focal developable of S_2 of revolution, i. e. two minimum cones with foci of meridian section for vertices.		
(12)(12) Two C_4 touching along two circles.			If $x_5 = \nu$, two C_2 having an element in common.		
			If $x_5 = \nu$, $x_2 = \lambda$, focal developable of cylinder which touches K.		
11(22). Two ruled cyclides, the double line is double directrix.			Two cylinders with minimum axis.		
1(23). Two ruled cyclides, the double line is single directrix and single generator.			Two parabolic cylinders with minimum axis.		
C_4 and quad.			*C_2 and quad.*		
(11)(11)2. The quad belongs to two non-consecutive generators of C_4.			C_2 and two finite minimum lines and line at infinity in their plane counted twice.		
			If $x_5 = \nu$, C_2 and quadrilateral inscribed in K.		
(11)(13). The quad lies on one generator of C_4.			If $x_1 + ix_2 = \nu$, C_2 and two minimum lines lying in same tangent plane.		
			If $x_4 = \nu$, $x_3 = \lambda$, focal developable of the paraboloid of revolution.		
			If $x_4 = \nu$, C_2 and line in Π.		
2(22). A ruled cyclide: the quadrilateral becomes two lines and a line cutting them counted twice.			A cone with vertex on K.		
(24). Same as above			Same as above.		

C_4 Counted Twice.	*C_2 Counted Twice.*
(111) 111. The complex consists of spheres which meet paired spheres of C_4 in an involution [2].	The complex consists of those spheres which touch paired planes and of a cone of revolution in involution [2].
(111)(11) 1. The complex consists of spheres which touch paired spheres of a C in an involution [2] having two pairs of double elements coincident.	As above, the involution having two pairs of coincident elements.
(111)(111). Complex consists of spheres which touch a Dupin cyclide.	Complex consists of spheres which touch a cone of revolution.
(111)(12). Same as (111) 111 involution has two coincident double elements.	Same as (111) 111, involution has two coincident double elements.
(111)(12). As above, involution has all its double elements coincident.	As above, the involution has all its double elements coincident.
(111) 3. Involution has three double elements coincident.	Involution has three double elements coincident.
(222). Complex consists of those spheres which touch a ruled cyclide.	Complex consists of those spheres which touch a cylinder which has a minimum axis.

SURFACES WHICH DEGRADE INTO MINIMUM LINES.

(11)(11)(11). Two skew quadrilaterals.	One finite quadrilateral and quadrilateral inscribed in K.
11(112). A quadrilateral counted twice. Complex formed by spheres which meet corresponding spheres of two pencils in (2, 2) correspondence having a self-corresponding sphere.	If $x_5 = \nu$, quadrilateral inscribed in K. If $x_1 + ix_2 = \nu$, one of the lines becomes tangent to K.
1(113). As above, the (2, 2) correspondence has three coincident double elements.	As above.

(11)(112). As above, the pencils are in (1, 1) correspondence and have a self-corresponding sphere.	As above.
(112) 2. As above. The pencils are in (i, 2) correspondence.	As above.
1 (122). Two intersecting minimum lines each counted four times.	A finite minimum line and a line at infinity which cuts it, each counted four times.
(11)(22). Two skew quadrilaterals having two common sides.	Two minimum lines and one of their common secants counted twice.
(114). Same as (112) 2.	Same as (112) 2.
(123). Two intersecting minimum lines counted four times.	If $x_2 = \nu$, quadrilateral in K counted twice. If $x_4 = \nu$, two finite intersecting minimum lines counted twice and line at infinity in their plane counted four times.
(33). Two minimum lines counted three times and two counted once.	One minimum line counted three times and one counted once.

CORNELL UNIVERSITY, *May* 1, 1904.

*Subgroups of Order a Power of p in the General and Special m-ary Linear Homogeneous Groups in the $GF[p^n]$.**

By Leonard Eugene Dickson.†

1. It would seem that the most effective method of determining all the subgroups of order a multiple of p of a linear group in the Galois field of order p^n is that based upon a complete knowledge of the subgroups of order a power of p. This method has proved successful for the ternary groups‡ and, as I will show on another occasion, also for the quaternary groups.

The present investigation proceeds far enough to give a clear insight into the nature of the simple laws pervading the subject. It is hoped that the results are capable of extension by induction to all powers of p. To indicate the difficulty of this step, it may be remarked that its completion would give the means of deriving at once an explicit list of all groups of order a power of a prime, and simultaneously all the subgroups of each.

A second aim of the paper was to furnish data for the problem of the determination of all m-ary groups for low values of m.

Following Lie, I write (A, B) for the commutator $A^{-1}B^{-1}AB$. I employ the usual notation $B_{ij\alpha}$ for the transformation which alters only ξ_i, replacing it by $\xi_i + \alpha\xi_j$. We have the simple relation §

$$(B_{it\alpha},\ B_{tj\beta}) = B_{ij-\alpha\beta}, \qquad (i > t > j), \qquad (1)$$

by use of which computations are reduced to a minimum.

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† Research assistant to the Carnegie Institution of Washington.

‡ American Journal of Mathematics, Vol. 27 (1905), pp. 189-202.

§ It follows from it that $SLH(m, p^n)$, $m > 2$, is a perfect group.

2. Let G be either the general or special linear homogeneous group, $GLH(m, p^n)$ or $SLH(m, p^n)$. The highest power of p dividing the order of G is $p^{\mu n}$, $\mu = \frac{1}{2} m (m-1)$. The totality of the transformations in the $GF[p^n]$

$$[a_{ij}]: \quad (a_{ij}) \text{ with every } a_{ij} = 0 \; (j > i), \quad a_{ii} = 1 \tag{2}$$

forms a subgroup $G_{p^{\mu n}}$ The law of composition is

$$\left.\begin{array}{l} [\alpha][\beta] = [\gamma], \\ \gamma_{ii-1} = \beta_{ii-1} + \alpha_{ii-1}, \quad \gamma_{ij} = \beta_{ij} + \alpha_{ij} + \sum\limits_{k=j+1}^{i-1} \beta_{ik}\alpha_{kj}, \qquad (j < i-1). \end{array}\right\} \tag{3}$$

Since every subgroup of order a power of p of G is conjugate with a subgroup of $G_{p^{\mu n}}$, it suffices to determine those of $G_{p^{\mu n}}$.

In case a subgroup of $G_{p^{\mu n}}$ can be defined by certain relations $r_1 = 0, \ldots, r_s = 0$ between the a_{ij}, we denote it $\{r_1 = 0, \ldots, r_s = 0\}$.

3. Let the $GF[p^n]$ be defined by an irreducible congruence

$$\rho^n \equiv \sum_{i=0}^{n-1} r_i \rho^i, \qquad (\text{mod } p). \tag{4}$$

Set

$$\alpha_{ij} = \sum_{k=0}^{n-1} a_{ijk}\rho^k, \quad \beta_{ij} = \sum_{k=0}^{n-1} b_{ijk}\rho^k, \quad \gamma_{ij} = \sum_{k=0}^{n-1} c_{ijk}\rho_k. \tag{5}$$

Relations (3) in the $GF[p^n]$ are equivalent to a set of congruences (3′) mod p, including

$$c_{ii-1k} \equiv b_{ii-1k} + a_{ii-1k}, \qquad (i = 2, \ldots, m; \; k = 0, \ldots, n-1). \tag{6}$$

If, for every pair of operators $[\alpha]$ and $[\beta]$ of a subgroup H of $G_{p^{\mu n}}$, $\phi(c) \equiv \phi(b) + \phi(a)$, we say that H is *additive with respect to* $\phi(a)$. Thus $G_{p^{\mu n}}$ is additive with respect to every a_{ii-1k}.

If H is additive with respect to a_{ijk}, then $[\alpha]^p = [\alpha']$ has $a'_{ijk} \equiv 0$, and $([\alpha], [\beta]) = [\delta]$ has $d_{ijk} \equiv 0$. In particular, all the commutators of $G_{p^{\mu n}}$ have every $d_{ii-1k} \equiv 0$.

4. LEMMA. *If the pth power of every operator of a group G_{p^g} belongs to its commutator subgroup G_{p^c}, there are exactly $(p^{g-c} - 1)/(p-1)$ subgroups of order p^{g-1} in G_{p^g}.*

Indeed, G_{p^g}/G_{p^e} is abelian of type $(1, 1, \ldots, 1)_{g-e}$ and contains every H/G_{p^e}, H being a subgroup of order p^{g-1}.

5. The commutator subgroup K of $G_{p^{\mu n}}$ is formed of the operators $[\delta_{ij}]$ with every $\delta_{ii-1} = 0$ (§3) and every δ_{ij} $(j < i - 1)$ arbitrary in the $GF[p^n]$. Indeed, in view of (1), K contains every $B_{ij\beta}$ $(j < i - 1)$ since each $B_{ij\alpha}$ $(j < i)$ belongs to $G_{p^{\mu n}}$. Then $[\alpha]^p$ belongs to K. Hence follows the

THEOREM. *The* $(p^{(m-1)n} - 1)/(p - 1)$ *subgroups of order* $p^{\mu n - 1}$ *of* $G_{p^{\mu n}}$ *are defined by a linear homogeneous relation between* a_{ii-1k} $(i = 2, \ldots, m;$ $k = 0, \ldots, n - 1)$.

For $n = 1$, we give to the subgroups the notation

$$\left\{ \sum_{i=2}^{m} c_i a_{ii-1} = 0 \right\}, \text{ not every } c_i \text{ zero}. \tag{7}$$

6. THEOREM.—*Let H be a subgroup defined by r independent linear homogeneous relations between the a_{ss-1k} $(k = 0, 1, \ldots, n - 1)$. If $r < n$, the $(p^{(m-1)n-r} - 1)/(p - 1)$ subgroups of index p under H are given by annexing any linear homogeneous relation between the a_{ii-1k} $(i = 2, \ldots, m;\ k = 0, \ldots, n - 1)$ independent of the r relations. If $r = n$, so that $H = \{a_{ss-1} = 0\}$, and $2 < s < m$, the $(p^{mn} - 1)/(p - 1)$ subgroups of index p under H are given by annexing any linear homogeneous relation between the a_{ii-1k} $(i \neq s)$, a_{ss-2k}, $a_{s+1s-1k}$; while for $s = 2$ or $s = m$, the $(p^{(m-1)n} - 1)/(p - 1)$ subgroups are obtained from the preceding by suppressing the a_{ss-2k} or $a_{s+1s-1k}$, respectively.*

H contains every $B_{ij\alpha}$ $(j < i)$ except certain $B_{ss-1\alpha}$. By (1),

$$(B_{ii-1\alpha}, B_{i-1i-2\beta}) = B_{ii-2\gamma}, \quad (B_{ii-2\alpha}, B_{i-2j\beta}) = B_{ij\gamma} \ (i - j > 3), \tag{8}$$

$$(B_{ii-1\alpha}, B_{i-1i-3\beta}) = B_{ii-3\gamma}, \quad (B_{ii-2\alpha}, B_{i-2i-3\beta}) = B_{ii-3\gamma}, \tag{9}$$

where $\gamma = -\alpha\beta$. By (8) the commutator group K of H contains every $B_{ii-2\gamma}$ $(i \neq s, s + 1)$ and every $B_{ij\gamma}$ $(i - j > 3)$. Applying $(9)_1$ if $i \neq s$ and $(9)_2$ if $i = s$, we find that K contains every $B_{ii-3\gamma}$.

If $r < n$, H contains a $B_{ss-1\alpha}$, $\alpha \neq 0$. By $(8)_1$ for $i = s$ and $i = s + 1$, K contains $B_{ss-2\gamma}$ and $B_{s+1s-1\gamma}$, where $\gamma = -\alpha\beta$ is arbitrary since β is. Hence K contains every $B_{ij\beta}$ $(j < i - 1)$.

If $r = n$, so that the relations require $\alpha_{ss-1} = 0$, H is additive with respect to α_{ss-2} and $\alpha_{s+1\,s-1}$ by (3). Hence K is composed of the operators $[\delta_{ij}]$ with every $\delta_{ii-1} = 0, \delta_{ss-2} = 0, \delta_{s+1\,s-1} = 0$, and the remaining δ_{ij} $(j < i)$ all arbitrary.

7. LEMMA. *If H is a subgroup defined by certain relations R between the a_{ii-1k} $(i = s, \ldots, s+t;\ k = 0, \ldots, n-1)$ not requiring that $\alpha_{ss-1} = 0$ if $s > 2$, nor that $\alpha_{s+t\,s+t-1} = 0$ if $s + t < m$, nor that $\alpha_{s+2\,s+1} = 0$ if $t > 1$, the commutator subgroup K of H contains every $B_{ij\alpha}$ $(j < i-1)$ except possibly $B_{ii-2\alpha}$ $(i = s+1, \ldots, s+t)$.*

H contains every $B_{ij\alpha}$ $(j < i)$ except $B_{ww-1\alpha}$ $(w = s, \ldots, s+t)$; also,

$$B_\alpha:\quad \xi_i' = \xi_i + \alpha_{ii-1}\xi_{i-1}, \qquad (i = s, \ldots, s+t), \tag{10}$$

the α_{ii-1} being subject to relations R. The inverse of B_α is

$$B_\alpha^{-1}:\quad \xi_i' = \xi_i + \sum_{j=1}^{i-s+1} (-1)^j \alpha_{ii-1}\alpha_{i-1\,i-2} \cdots \alpha_{i-j+1\,i-j}\xi_{ij}. \qquad (i = s, \ldots, s+t). \tag{11}$$

In view of (8) and (9), K contains every

$$B_{ij\gamma}(i - j > 3),\ B_{ii-3\gamma}(i \neq s, \ldots, s+t+1),\ B_{ii-3\gamma}(i \neq s+2, \ldots, s+t). \tag{12}$$

If $s > 2$, $(B_{s-1\,s-2\tau}, B_\alpha) = B_{ss-2\sigma}$, where $\sigma = \tau\alpha_{ss-1}$ may be made arbitrary by choice of τ. If $s + t < m$,

$$(B_{s+t+1\,s+t\tau}, B_\alpha):\ \xi'_{s+t+1} = \xi_{s+t+1} + \tau\sum_{j=1}^{t+1}(-1)^j\alpha_{s+t\,s+t-1}\cdots\alpha_{s+t-j+1\,s+t-j}\xi_{s+t-j}.$$

Multiplying this on the right by $\prod_{j=2}^{t+1} B_{s+t+1\,s+t-j\lambda_j}$, which belongs to K, we obtain $B_{s+t+1\,s+t-1\lambda}$ by suitably choosing the λ_j; here $\lambda = -\tau\alpha_{s+t\,s+t-1}$ may be made arbitrary. If $t > 1$ and $s + 1 \leqq l \leqq s + t - 1$,

$$C_l \equiv (B_{ll-2\tau}, B_\alpha):\ \xi'_l = \xi_l + \tau f,\ \xi'_{l+1} = \xi_{l+1} + \tau\alpha_{l+1\,l}\,\xi_{l-2} + \tau\alpha_{l+1\,l}f,$$

$$f \equiv \sum_{j=1}^{l-s-1}(-1)^j\alpha_{l-2\,l-3}\cdots\alpha_{l-j-1\,l-j-2}\xi_{l-j-2}.$$

In particular, $C_{s+1} = B_{s+2s-1\kappa}$, where $\kappa = \tau a_{s+2s+1}$ may be made arbitrary. Then by choice of λ_1 and λ_2,

$$C_{s+2} B_{s+2s-1\lambda_1} B_{s+3s-1\lambda_2} = B_{s+3s\lambda},$$

where λ is arbitrary. Next, by choice of μ_1, μ_2, ν_1, ν_2,

$$C_{s+3} B_{s+3s\mu_1} B_{s+3s-1\mu_2} B_{s+4s\nu_1} B_{s+4s-1\nu_2} = B_{s+4s+1\nu},$$

where ν is arbitrary. It follows in this way that K contains every $B_{ii-3\gamma}$ $(i = s+2, \ldots, s+t)$. But these with $B_{ss-2\sigma}$, $B_{s+t+1s+t-1\lambda}$ and (12), give all stated in the lemma.

8. THEOREM.—*Let K be the commutator subgroup of*

$$\left\{\sum_{i=s}^{s+t} c_i a_{ii-1} = 0\right\}, c_s \neq 0,\ c_{s+t} \neq 0,\ t \geqq 1. \tag{13}$$

If $t = 1$, K is composed of the operators $[\delta_{ij}]$, $\delta_{ii-1} = 0$, $\delta_{s+1s-1} = 0$, the remaining δ_{ij} $(j < i-1)$ being arbitrary. If $t = 2$, then $\delta_{ii-1} = 0$, while the δ_{ij} $(j < i-1)$ are subject only to the condition.

$$c_s \delta_{s+1s-1} - c_{s+2} \delta_{s+2s} = 0. \tag{14}$$

If $t \geqq 3$, then $\delta_{ii-1} = 0$, while every δ_{ij} $(j < i-1)$ is arbitrary.

By §7, K contains every $B_{ij\kappa}$ $(j < i-1)$ except $B_{ii-2\kappa}$ $(i = s+1, \ldots, s+t)$. Next, K contains

$$(B_\alpha, B_\beta):\ \xi_i' = \xi_i + \sum_{j=i-2}^{s-1} \delta_{ij} \xi_j \qquad (i = s+1, \ldots, s+t) \tag{15}$$

where

$$\delta_{ii-2} = \beta_{ii-1} \alpha_{i-1i-2} - \alpha_{ii-1} \beta_{i-1i-2}. \tag{16}$$

By choice of the ρ_{ij} the product of (15) by $\prod_{i=s+2}^{s+t} \prod_{j=i-3}^{s-1} B_{ij\rho_{ij}}$ becomes

$$\xi_i' = \xi_i + \delta_{ii-2} \xi_{i-2} \qquad (i = s+1, \ldots, s+t). \tag{17}$$

Hence K contains every operator (17) subject to (16). Set

$$c_j' = c_{s+j},\ \alpha_j = \alpha_{s+js+j-1},\ \beta_j = \beta_{s+js+j-1},\ \delta_j = \delta_{s+js+j-2}\ (j = 1, \ldots, t). \tag{18}$$

Then (16), (13), and $\Sigma c_i \beta_{ii-1} = 0$ become, respectively,*

$$\delta_j = \beta_j \alpha_{j-1} - \alpha_j \beta_{j-1} (j = 1, \ldots, t), \sum_{j=0}^{t} c'_j \alpha_j = 0, \sum_{j=0}^{t} c'_j \beta_j = 0. \tag{19}$$

Since $c'_0 \neq 0$, the last two equations serve to determine α_0 and β_0. Eliminating the latter from δ_1, we get

$$c'_0 \delta_1 = \sum_{i=2}^{t} (\alpha_1 \beta_i - \beta_1 \alpha_i) c'_i.$$

Hence $\delta_1 = 0$ if $t = 1$. For $t > 1$, $c'_0 \delta_1 - c'_2 \delta_2$ equals

$$\delta \equiv \sum_{i=3}^{t} (\alpha_1 \beta_i - \beta_1 \alpha_i) c'_i. \tag{20}$$

Hence $\delta = 0$ if $t = 2$, and we can make either δ_1 or δ_2 arbitrary, the other being determined by $c'_0 \delta_1 - c'_2 \delta_2 = 0$, viz. (14).

If $t > 2$, we proceed to show that $\delta, \delta_2, \ldots, \delta_t$ (and hence $\delta_1\ \delta_2, \ldots, \delta_t$) can be made arbitrary by choice of the α_i, β_i $(i = 1, \ldots, t)$. Now $\delta_2, \ldots, \delta_t - 1$ involve neither β_t nor α_t, while in δ and δ_t the determinant of the coefficients of β_t and α_t is $-c'_t \Delta$, where

$$\Delta \equiv \alpha_1 \beta_{t-1} - \beta_1 \alpha_{t-1}.$$

Hence it suffices to make $\Delta \neq 0$ and $\delta_2, \ldots, \delta_{t-1}$ arbitrary.

For $t \parallel 2\tau$, $\tau > 1$, it suffices to take

$$\beta_1 = 0, \beta_{2j-1} = 1 (j = 2, 3, \ldots, \tau), \alpha_1 = 1, \alpha_{4l-1} = 1, \alpha_{4l+1} = 0 \, (l = 1, 2, \ldots).$$

Then

$$\Delta = 1,\ \delta_2 = \beta_2,\ \delta_{4j+2} = -\alpha_{4j+2},\ \delta_{4j+1} = \alpha_{4j},$$
$$\delta_{4j} = \beta_{4j} - \alpha_{4j}\ (j > 0),\ \delta_{4j+3} = \alpha_{4j+2} - \beta_{4j+2}.$$

For $t = 2\tau + 1$, $\tau > 1$, it suffices to take

$$\beta_1 = 0,\ \alpha_1 = 1,\ \beta_{2j} = 1 (j = 2, \ldots, \tau), \alpha_{4j} = 0\ (j > 0),\ \alpha_{4j+2} = 1.$$

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* The determinant of the coefficients of $\alpha_0, \alpha_1, \ldots, \alpha_t$ equals

$$\left(\prod_{j=1}^{t-1} \beta_j\right) \sum_{i=0}^{t} c'_i \beta_i = 0.$$

Then

$$\Delta = 1,\ \delta_2 = \beta_2,\ \delta_4 = \alpha_3,\ \delta_3 = \beta_3 - \alpha_2\beta_2,\ \delta_{4j} = \alpha_{4j-1},$$
$$\delta_{4j+1} = -\alpha_{4j+1},\ \delta_{4j+2} = \alpha_{4j+1} - \beta_{4j+1},\ \delta_{4j+3} = \beta_{4j+3} - \alpha_{4j+2}\ (j > 0).$$

Finally, for $t = 3$, $\Delta = \delta_2$. If $\delta_2 \neq 0$, it suffices to take $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = \delta_2$. If $\delta_2 = 0$, $\delta_3 \neq 0$, it suffices to take

$$\beta_1 = \beta_2 = 0,\ \alpha_2 = 1,\ \beta_3 = \delta_3,\ \alpha_1\delta_3 c_3' = \delta.$$

If $\delta_2 = \delta_3 = 0$, it suffices to take $\beta_1 = \beta_2 = \alpha_2 = 0$, $\beta_3 = 1$, $\alpha_1 c_3' = \delta$.

9. For $t = 1$, $\delta_{s+1\,s-1} = 0$ gives* $\beta_{s+1\,s}\alpha_{ss-1} = \alpha_{s+1\,s}\beta_{ss-1}$. Thus, if $p > 2$, (13) is additive with respect to $\alpha_{s+1\,s-1} - \frac{1}{2}\alpha_{s+1\,s}\alpha_{ss-1}$. Hence $[\alpha]^p = [x]$ has every $x_{ii-1} = 0$, $x_{s+1\,s-1} = 0$, and thus belongs to K. *For* $p > 2$, $n = 1$, *the* $(p^{m-1} - 1)/(p - 1)$ *subgroups of order* $p^{\mu-2}$ *of* $\{c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1\,s} = 0\}$, $c_s \neq 0$, $c_{s+1} \neq 0$, *are*

$$\left\{c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1\,s} = 0,\ k_1(\alpha_{s+1\,s-1} - \tfrac{1}{2}\alpha_{s+1\,s}\alpha_{ss-1}) + \sum_{\substack{i=2\\ i\neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \tag{21}$$

10. For $t = 2$, $p > 2$, group (13) is additive with respect to

$$f_\alpha \equiv c_s\alpha_{s+1\,s-1} - c_{s+2}\alpha_{s+2\,s} - \tfrac{1}{2}c_s\alpha_{s+1\,s}\alpha_{ss-1} + \tfrac{1}{2}c_{s+2}\alpha_{s+2\,s+1}\alpha_{s+1\,s}. \tag{22}$$

Indeed, employing the values of γ_{ii-1}, $\gamma_{s+1\,s-1}$, $\gamma_{s+2\,s}$ from (3), we get

$$f_\gamma - f_\alpha - f_\beta = \tfrac{1}{2}(c_s\delta_{s+1\,s-1} - c_{s+2}\delta_{s+2\,s}) = 0.$$

Further, if $[\alpha]^p$ be written $[\delta]$, then each $\delta_{ii-1} = 0$ and (14) holds, so that it belongs to K. *For* $t = 2$, $p > 2$, $n = 1$, *the* $(p^{m-1} - 1)/(p - 1)$ *subgroups of order* $p^{\mu-2}$ *of* (13) *are*

$$\left\{\sum_{i=s}^{s+2} c_i\alpha_{ii-1} = 0,\ k_1 f_\alpha + \sum_{\substack{i=2\\ i\neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \tag{23}$$

11. *For* $t > 2$, $n = 1$, *the* $(p^{m-2} - 1)/(p - 1)$ *subgroups of order* $p^{\mu-2}$ *of* (13) *are*

$$\left\{\sum_{i=s}^{s+t} c_i\alpha_{ii-1} = 0,\ \sum_{\substack{i=2\\ i\neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \tag{24}$$

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* Or directly from $c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1\,s} = 0$, $c_s\beta_{ss-1} + c_{s+1}\beta_{s+1\,s} = 0$, $c_s \neq 0$, $c_{s+1} \neq 0$.

12. For $n=1, p>2$, all the subgroups of order $p^{\mu-2}$ of G_{p^μ} have been determined in §§6, 9, 10, 11. The distinct ones are (21) and (23) for $k_1=1$, and

$$\left\{\sum_{i=2}^{m} c_i a_{i\,i-1}=0,\ \sum_{i=2}^{m} k_i a_{i\,i-1}=0\right\},\quad D_{ij}\equiv\begin{vmatrix} c_i & c_j \\ k_i & k_j \end{vmatrix}\ \text{not all zero,} \tag{25}$$

$$\left\{a_{s\,s-1}=0,\ \sum_{\substack{i=2,\dots,m\\ i\neq s}} k_i a_{i\,i-1}+k_s a_{s\,s-2}+k_1 a_{s+1\,s-1}=0\right\},\quad k_s, k_1\ \text{not both } 0, \tag{26}$$

where $k_s a_{s\,s-2}$ is to be suppressed if $s=2$, and $k_1 a_{s+1\,s-1}$ if $s=m$.

13. If $k_{s-1}=0$ in (26), we transform by

$$T_{s-1\,s}:\quad \xi'_{s-1}=\xi_s,\ \xi'_s=-\xi_{s-1}, \tag{27}$$

and get a group of the form (26) with $k_s=0$. If $k_{s-1}\neq 0$, the same result follows by transforming (26) by $B_{s-1\,s\,\rho}$, $\rho=k_s/k_{s-1}$. Consider (26) with $k_s=0$, $k_1=-1$, as we may set. If also $k_{s-1}=0$, $k_{s+1}\neq 0$, we transform by $B_{s-1\,s\,\rho}$, $\rho=k_{s+1}^{-1}$, and obtain a group (25). If $k_{s-1}=k_{s+1}=0$, we transform by (27) and obtain a group (25). If $k_{s-1}\neq 0$, we may make $k_{s-1}=1$ by transforming by

$$\xi'_{s-2}=k_{s-1}^{-1}\xi_{s-2},\ \xi'_s=k_{s-1}\xi_s.$$

Then transforming by $B_{s\,s-1\,\rho}$, $\rho=k_{s+1}$, we obtain

$$\left\{a_{s\,s-1}=0,\ a_{s+1\,s-1}=\sum_{\substack{i=2,\dots,m\\ i\neq s,\,s+1}} x_i a_{i\,i-1}\right\},\ x_{s-1}=1. \tag{28}$$

If $s=2$, $a_{s-1\,s-2}$ is to be suppressed. If $s=m$, $a_{s+1\,s-1}$ is to be replaced by zero and the group falls under (25).

In (21) or (23) with $k_1=1$, we may make $k_{s+1}=0$ by transforming by $B_{s\,s-1\,\rho}$, so that f_α is replaced by $f_\alpha-\rho c_s a_{s+1\,s}$.

14. Consider the commutator subgroup K of the group H defined by (28) for $s<m$. H contains $B_{s+1\,s\,\alpha}$, $B_{i\,i-3\,\alpha}$ $(i\neq s+1)$, $B_{ij\alpha}$ $(j<i-2)$,

$$C_\alpha\equiv B_{s+1\,s-1\,\alpha}B_{s-1\,s-2\,\alpha},\ A_{i\alpha}\equiv B_{i\,i-1\,\alpha}B_{s+1\,s-1\,\kappa_i\alpha},\ B_{i\alpha}\equiv B_{s-1\,s-2\,-\kappa_i\alpha}B_{i\,i-1\,\alpha},$$

for $i\neq s-1, s, s+1$. Hence K contains

$(B_{i\,i-3\,\alpha}, B_{i-3\,j\,\beta})=B_{ij\gamma}$ $(j\leqq i-6)$, $(B_{i\,i-2\,\alpha}, B_{i-2\,i-4\,\beta})=B_{i\,i-4\,\gamma}$ $(i\neq s+1, s+3)$,
$(B_{i\,i-3\,\alpha}, B_{i-3\,i-5\,\beta})=B_{i\,i-5\,\gamma}$ $(i\neq s+4)$, $(B_{i\,i-2\,\alpha}, B_{i-2\,i-5\,\beta})=B_{i\,i-5\,\gamma}$ $(i\neq s+1)$,

$\gamma = -\alpha\beta$. By the last two, K contains every $B_{ii-5\gamma}$. Also

$$(B_{s-1s-3\rho},\, C_\alpha) = B_{s+1s-3\alpha\rho},\ (B_{s+3s+1\rho},\, C_\alpha) = B_{s+3\ s-1-\alpha\rho} B_{s+3s-2\alpha^2\rho},$$
$$(A_{i\alpha},\, B_{i-1i-3\beta}) = B_{ii-3\gamma} (i \neq s-1, s, s+1, s+2, s+4),$$
$$(A_{s-2\alpha},\, B_{ss-2\beta}) = B_{ss-3\alpha\beta},$$
$$(B_{s+1s\alpha},\, B_{ss-2\beta}) = B_{s+1s-2\gamma},\ (B_{s-2s-4\alpha},\, C_1) = B_{s-1s-4\alpha},$$
$$(A_{s+4\alpha},\, B_{s+3s+1\beta}) \colon \quad \xi'_{s+4} = \xi_{s+4} - \alpha\beta\xi_{s+1} + \varkappa\alpha^2\beta\xi_{s-1},$$
$$\xi'_{s+3} = \xi_{s+3} + \varkappa\alpha\beta\xi_{s-1},$$
$$(A_{s+2\alpha},\, C_1) \colon \quad \xi'_{s+2} = \xi_{s+2} - \alpha\xi_{s-1} + \alpha\xi_{s-2},\ \xi'_{s+1} = \xi_{s+1} - \varkappa\alpha\xi_{s-3}.$$

Hence K contains every $B_{ii-4\gamma}$, $B_{ii-3\gamma}$. Also

$$(B_{i\alpha},\, B_{i-11}) = B_{ii-2-\alpha}\ (i \neq s-2, s-1, s, s+1),$$
$$(A_{s-2\alpha},\, A_{s-31}) = B_{s-2s-4-\alpha}$$
$$(C_\alpha,\, A_{s-21}) = \xi'_{s+1} = \xi_{s+1} + \varkappa\alpha\xi_{s-2} - \varkappa\alpha\xi_{s-3},\ \xi'_{s-1} = \xi_{s-1} - \alpha\xi_{s-3}.$$

Hence K contains every $B_{ii-2\gamma}$ $(i \neq s, s+1)$. But in every transformation $[\delta_{ij}]$ of K, $\delta_{ii-1} = 0$, $\delta_{ss-2} = 0$, $\delta_{s+1s-1} = 0$. We have shown that the remaining δ_{ij} $(j < i)$ may be chosen arbitrarily. Further, $[\alpha]^p$ belongs to K. Also, H is additive with respect to α_{ii-1}, α_{ss-2}, α_{s+1s-1}.

THEOREM. *For* $n = 1$, $2 < s < m$, *the* $(p^{m-1} - 1)/(p-1)$ *subgroups of index* p *of* (28) *are given by annexing a linear relation between* α_{ss-2}, $\alpha_{ii-1} (i = 2, \ldots, m;\ i \neq s)$. *For* $s = 2$, *the* $(p^{m-2} - 1)/(p-1)$ *subgroups are given by annexing a linear relation between the* $\alpha_{ii-1} = 0$ $(i \neq s)$.

15. We may show similarly that the commutator subgroup of (23) with $k_1 = 1$, $k_{s+1} = 0$ (see end of §13) is composed of the $[\delta_{ij}]$ with $\delta_{ii-1} = 0$, and the δ_{ij} $(j < i-1)$ subject only to (14), and that it contains every $[\alpha]^p$. Hence, if $n = 1$, there are $(p^{m-2} - 1)/(p-1)$ subgroups of index p given by annexing a linear relation between the α_{ii-1}.

The commutator subgroup of (21) with $k_1 = 1$, $k_{s+1} = 0$ has $\delta_{ii-1} = 0$, $\delta_{s+1s-1} = 0$ and the remaining δ_{ij} $(j < i-1)$ arbitrary. Hence, if $n = 1$, there are $(p^{m-2} - 1)/(p-1)$ subgroups of index p given by annexing a linear relation between the α_{ii-1}.

I have not completed the longer discussion necessary for (25).

16. In the general problem special treatment is necessary for

$$\{\alpha_{s_1 s_1-1}=0,\ \alpha_{s_2 s_2-1}=0,\ \ldots,\ \alpha_{s_r s_r-1}=0\},\ s_1<s_2<\ldots<s_r. \qquad (29)$$

The commutator group K of (29) is formed of the operators $[\delta_{ij}]$ with $\delta_{ii-1}=0$, $\delta_{jj-2}=0$, where j ranges over the distinct numbers of the set

$$s_1,\ s_2,\ \ldots,\ s_r,\ s_1+1,\ s_2+1,\ \ldots,\ s_r+1, \qquad (30)$$

$\delta_{ii-3}=0$ $(i=\sigma_1,\ \ldots,\ \sigma_\rho)$, where $\sigma_1,\ \ldots,\ \sigma_\rho$ denote the distinct numbers such that both σ_i and σ_i-2 belong to the set $s_1,\ \ldots,\ s_r$, while the remaining δ_{ij} are arbitrary. In proof, (29) contains every $B_{ij\alpha}(j<i)$ except $B_{s_i s_i-1\alpha}$ $(i=1,\ \ldots,\ r)$. Then by (8), K contains every $B_{ij\gamma}$ $(j<i-3)$ and every $B_{ii-2\gamma}$, i not in the set (30). By (9), K contains every $B_{ii-3\gamma}$, $i\neq\sigma_1,\ \ldots,\ \sigma_\rho$. Moreover, (29) is additive with respect to the α_{jj-2}, j in (30), and the α_{ii-3}, $i=\sigma_1,\ \ldots,\ \sigma_\rho$. Hence K has the form given. It follows that, *if* $n=1$, *the subgroups of index* p *of* (29) *are obtained by annexing a linear relation between the* α_{ii-1} $(i\neq s_1,\ \ldots,\ s_r)$, *the* α_{jj-2}, j *in the set* (30), *and* α_{ii-3} $(i=\sigma_1,\ \ldots,\ \sigma_\rho)$.

Subgroups of order a power of p of SLH $(3, p^n)$.

17. We consider the subgroups of $G_{p^{3n}}$. The commutator

$$([\alpha],\ [\beta])=B_{3,1,\delta},\ \delta=\beta_{32}\alpha_{21}-\alpha_{32}\beta_{21}.$$

By §5, each of the $(p^{3n}-1)/(p-1)$ subgroups of order p^{3n-1} is defined by

$$f_1(\alpha)=\sum_{k=0}^{n-1}(\lambda_k\alpha_{21k}+\mu_k\alpha_{32k})\equiv 0\ (\text{mod}\ p).$$

If $n>1$, the commutator subgroup of any of these $G_{p^{3n-1}}$ is formed of the p^n operators $B_{3,1,\delta}$. In proof, we show that δ may be made arbitrary. If $\lambda_i\neq 0$, we take $\alpha_{32}=0$, $\alpha_{21j}=1$, j being a particular subscript $\neq i$. Then $\alpha_{21}\neq 0$ in view of the irreducibility of (4). Hence we can choose β_{32} to make δ arbitrary and then determine β_{21} to make $f_1(b)\equiv 0$. If every $\lambda_k=0$, we take $\beta_{32}=0$, the α_{32k} such that $\alpha_{32}\neq 0$ and $f_1(\alpha)=0$, and determine β_{21} to make δ arbitrary. *Hence, if* $n>1$, *the* $(p^{2n-1}-1)/(p=1)$ *subgroups of index* p *of* $G_{p^{3n-1}}$ *are given by*

$$f_1(\alpha)\equiv 0,\ \ f_2(\alpha)\equiv 0.$$

Consider such a subgroup $G_{p^{3n-2}}$ for $n > 2$. If the determinant of the coefficients of a_{21i} and a_{21j} in $f_1(a)$ and $f_2(a)$ is not $\equiv 0 \pmod p$, we take $\alpha_{33} = 0$, $a_{21l} = 1$, l being a particular subscript different from i and j; then $\alpha_{31} \neq 0$ and we may determine β_{32} to make δ arbitrary in the $GF[p^n]$, and b_{21i} and b_{21j} to make $f_1(b) \equiv 0$, to $f_2(b) \equiv 0$. A similar result follows if the determinant of the coefficients of a_{32i} and a_{32j} is not $\equiv 0 \pmod p$. There remains the case

$$\left\{\sum_{k=0}^{n-1} \lambda_k a_{21k} \equiv 0, \quad \sum_{k=0}^{n-1} l_k a_{32k} \equiv 0\right\}.$$

Now $\xi_1' = s^{-1}\xi_1$, $\xi_3' = t\xi_3$ transforms $[\alpha]$ into $[\alpha']$ where

$$\alpha_{21}' = s\alpha_{21}, \quad \alpha_{32}' = t\alpha_{32}, \quad \alpha_{31}' = st\alpha_{31}.$$

Let $s = \Sigma s_k \rho^k$, $\alpha_{21}' = \Sigma a_{21k}' \rho^k$. Then a_{21k}' is a bi-linear form in s_k, a_{21k} $(k = 0, 1, \ldots, n-1)$ of determinant* not $\equiv 0 \pmod p$. Hence we may choose s so that a_{21n-1}' shall be identical with $\Sigma \lambda_k a_{21k}$. Proceeding similarly with α_{32}', the group becomes $\{a_{21n-1} = a_{32n-1} = 0\}$. For the latter we take $a_{210} = 1$, $a_{21k} = 0\ (k > 0)$, $b_{210} = 0$, $b_{211} = 1$, $b_{21k} = 0\ (k > 1)$, $b_{32k} = 0\ (k > 0)$. Then

$$\delta = b_{320} - \sum_{k=0}^{n-2} a_{32k} \rho^{k+1}$$

is arbitrary in the $GF[p^n]$. *Hence the commutator subgroup of $G_{p^{3n-2}}$ for $n > 2$ is of order p^n; the $(p^{2n-2} - 1)/(p - 1)$ subgroups of index p of $G_{p^{3n-2}}$ are defined by three linear congruences $f_1(a) \equiv 0$, $f_2(a) \equiv 0$, $f_3(a) \equiv 0$.*

Consider such a subgroup $G_{p^{3n-3}}$ for $n > 3$. If the determinant of the coefficients of a_{21i}, a_{21j}, a_{21l} is not $\equiv 0 \pmod p$, we take $\alpha_{33} = 0$, $a_{21r} \equiv 1$, r being a particular subscript different from i, j, l; then $\alpha_{21} \neq 0$, and we can determine β_{32} to make δ arbitrary in the $GF[p^n]$ and b_{21i}, b_{21j}, b_{21l} to make $f_1(b) \equiv 0$, $f_2(b) \equiv 0$, $f_3(b) \equiv 0$. If all such determinants are $\equiv 0 \pmod p$ and likewise for the determinants of the coefficients of a_{32i}, the three relations become

$$\sum_{k=0}^{n-1} m_k a_{21k} \equiv 0, \sum_{k=0}^{n-1} l_k a_{32k} \equiv 0, f_3(a) \equiv 0.$$

As before we normalize by transformation and reach

$$\left\{a_{21n-1} \equiv 0, \ a_{32n-1} \equiv 0, \ f_3(a) = \sum_{k=0}^{n-2} (\lambda_k a_{21k} + \mu_k a_{32k}) \equiv 0\right\}.$$

* *A general theorem on algebraic numbers*, Bull. Amer. Math. Soc., vol. 11 (1905), pp. 482-486.

We proceed to show that in $\delta = \Sigma d_k \rho^k$ the d_k may be made arbitrary mod p. In view of the symmetry, we may assume that the μ_k are not all zero. We first take $\alpha_{21} = 1$, $\beta_{21} = \rho$. Then

$$d_0 = b_{320},\ d_{n-1} = -a_{32n-2},\ d_k = b_{32k} - a_{32k-1}\ (k = 1, \ldots, n-2).$$

The conditions $f_3(a) \equiv 0$, $f_3(b) \equiv 0$ become, respectively,

$$\begin{array}{cccc|l} a_{320} & a_{321} & a_{322} \cdots & a_{32n-3} & \\ \hline \mu_0 & \mu_1 & \mu_2 \cdots & \mu_{n-3} & = -\lambda_0 + \mu_{n-2} d_{n-1} \\ \mu_1 & \mu_2 & \mu_3 \cdots & \mu_{n-2} & = -\lambda_1 - \mu_0 d_0 - \sum_{k=1}^{n-2} \mu_k d_k . \end{array}$$

They may be satisfied by choice of the a_{32k} unless every

$$\begin{vmatrix} \mu_i & \mu_j \\ \mu_{i+1} & \mu_{j+1} \end{vmatrix} \equiv 0 \qquad (i, j = 0, 1, \ldots, n-3). \tag{31}$$

We next take $\alpha_{21} = 1$, $\beta_{21} = \rho^2$. Then, applying (4),

$d_0 = b_{320} - a_{32n-2} r_0$, $d_1 = b_{321} - a_{32n-2} r_1$,

$d_{n-1} = -a_{32n-3} - a_{32n-2} r_{n-1}$, $d_k = b_{32k} - a_{32k-2} - a_{32n-2} r_k$ $(k = 2, \ldots, n-2)$.

Conditions $f_3(a) \equiv 0$, $f_3(b) \equiv 0$ become, respectively,

$$\begin{array}{ccc|l} a_{320} & a_{321} \cdots a_{32n-4} & a_{32n-2} & \\ \hline \mu_0 & \mu_1 \cdots \mu_{n-4} & \mu_{n-2} - \mu_{n-3} r_{n-1} & = -\lambda_0 + \mu_{n-3} d_{n-1} \\ \mu_2 & \mu_3 \cdots \mu_{n-2} & \sum_{k=0}^{n-2} \mu_k r_k & = -\lambda_2 - \sum_{k=0}^{n-2} \mu_k d_k . \end{array}$$

They may be satisfied by choice of the a_{32k} unless every

$$\begin{vmatrix} \mu_i & \mu_j \\ \mu_{i+2} & \mu_{j+2} \end{vmatrix} \equiv 0, \quad D_i = \begin{vmatrix} \mu_i & \mu_{n-2} - \mu_{n-3} r_{n-1} \\ \mu_{i+2} & \sum_{k=0}^{n-2} \mu_k r_k \end{vmatrix} \equiv 0, \tag{32}$$

for $i, j = 0, 1, \ldots, n-4$. Let, therefore, (31) and (32) all hold. If $\mu_0 \equiv 0$, then $\mu_1 \equiv 0, \ldots, \mu_{n-3} \equiv 0$ by (31) and $\mu_{n-2} \equiv 0$ by $D_{n-4} \equiv 0$, whereas not every $\mu_k \equiv 0$ by hypothesis. Hence $\mu_0 \not\equiv 0$. Then in $f_3(a) \equiv 0$, we may set $\mu_0 \equiv 1$. Then by (31), $\mu_k \equiv \mu_1^k$ $(k = 1, \ldots, n-2)$. Then

$$D_0 \equiv \sum_{k=0}^{n-1} \mu_1^k r_k - \mu_1^n \equiv 0,$$

so that (4) has the root $\rho = \mu_1$, contrary to its irreducibility. Hence, if $n > 3$, the commutator subgroup of $G_{p^{3n-3}}$ is of order p^n. *For* $n > 3$, *the* $(p^{2n-3} - 1)/(p - 1)$ *subgroups of index* p *of* $G_{p^{3n-3}}$ *are obtained by annexing a fourth linear relation* $f_4(a) \equiv 0$.

I do not enter upon further details here of the proof of the theorem: *For* $r \overline{\gtrless} n$, *any subgroup of order* p^{3n-r} *of* $G_{p^{3n}}$ *is defined by* r *independent linear homogeneous congruences between the* a_{31k}, a_{32k} $(k = 0, 1, \ldots, n-1)$.

18. I will here express my belief in the truth of the following general theorem: *For* $r \overline{\gtrless} n$, *any subgroup of order* $p^{\mu n - r}$ *of* $G_{p^{\mu n}}$ *is defined by* r *independent linear homogeneous congruences between the* $a_{i,i-1,k}$ $(i = 2, \ldots, m;$ $k = 0, 1, \ldots, n-1)$. Such a theorem would include the results of §§5, 6, 17, 20.

19. We proceed to determine the subgroups of order p^3 of $SLH(3, p^2)$. For (4) we take $\rho^2 \equiv \mu\rho + \nu \pmod{p}$. Set $\delta = d + D\rho$. Then

$$d \equiv \Delta_{00} + \nu\Delta_{11},\ D \equiv \Delta_{01} + \Delta_{10} + \mu\Delta_{11},\ \Delta_{ij} = b_{32i}a_{21j} - a_{32i}b_{21j}.$$

Any subgroup G_{p^4} of G_{p^6} is defined (§17) by two independent congruences $f_1(a) \equiv 0$, $f_2(a) \equiv 0$. If the determinant of the coefficients of a_{210} and a_{211} is $\not\equiv 0$, we get

$$\{a_{210} \equiv g a_{320} + h a_{321},\ a_{211} \equiv l a_{320} + m a_{321}\}. \tag{33}$$

In the contrary case, we obtain one of the following:

$$\{a_{321} = h a_{320},\ a_{211} = l a_{210} + m a_{320}\},\ \{a_{321} = h a_{320},\ a_{210} = m a_{320}\}, \tag{34}$$

$$\{a_{320} = 0,\ a_{211} = l a_{210} + m a_{321}\},\ \{a_{320} = 0,\ a_{210} = m a_{321}\},\ \{a_{32} = 0\}. \tag{35}$$

For the five groups (33) - (35), d and D are respectively

$$(h - l\nu)(b_{320}a_{321} - a_{320}b_{321}),\ (m - g - l\mu)(b_{320}a_{321} - a_{320}b_{321}); \tag{33'}$$

$$(1 + hl\nu)\Delta_{00},\ (h + l + hl\mu)\Delta_{00};\ h\nu\Delta_{01},\ (1 + h\mu)\Delta_{01}; \tag{34'}$$

$$l\nu\Delta_{10},\ (1 + l\mu)\Delta_{10};\ \nu\Delta_{11},\ \mu\Delta_{11};\ 0,\ 0. \tag{35'}$$

The a's and b's entering (33)'-(35)' are arbitrary. Now d and D are both identically zero only for $\{a_{32} = 0\}$ and for (33) with the two coefficients in (33)' vanishing so that (33) becomes

$$\{a_{21} = \tau a_{32}\},\quad \tau = g + \rho h \nu^{-1}. \tag{36}$$

The statement is evident except for $(34)_1$. But if $1 + hl\nu = 0$ and $h + l + hl\mu = 0$, then $-h^{-1}$ and $-l^{-1}$ are the roots of $\rho^2 - \mu\rho - \nu \equiv 0$, contrary to the irreducibility of (4). *The only commutative* G_{p^4} *are* $\{\alpha_{32} = 0\}$ and $\{\alpha_{21} = \tau\alpha_{32}\}$; *the commutator subgroups of the remaining* G_{p^4} *are of order* p.

If $p > 2$, every ternary $[\alpha]$ is of period p. The number of subgroups G_{p^3} of each G_{p^4} now follows from §4. They may be obtained by employing (33′)-(35)′.

If $p > 2$, *the subgroups* G_{p^3} *of the* G_{p^4} *given in the first column are obtained by annexing to the two relations defining the* G_{p^4} *an independent linear homogeneous relation between the elements given in the same line of the second column*:

$(33) = (36)$	$a_{320},\ a_{321},\ a_{310} - \frac{1}{2}ga_{320}^2 - ha_{320}a_{321} - \frac{1}{2}(\nu g + \mu h)\,a_{321}^2,$
	$a_{311} - \frac{h}{2\nu}a_{320}^2 - (g + \frac{\mu h}{\nu})\,a_{320}a_{321} - \frac{1}{2}(h + \frac{h\mu^2}{\nu} + \mu g)\,a_{321}^2$
$(33) \neq (36)$	$a_{320},\ a_{321},\ (m - g - l\mu)\,a_{310} - (h - \nu l)\,a_{311} - ua_{320}^2 - va_{320}a_{321} - wa_{321}^2$
$(34)_1$	$a_{320},\ a_{210},\ (l + h + lh\mu)\,a_{310} - (1 + hl\nu)\,a_{311} + \frac{m}{2}(1 + h\mu - h^2\nu)\,a_{320}^2$
$(34)_2$	$a_{320},\ a_{211},\ (1 + h\mu)\,a_{310} - h\nu a_{311} - \frac{m}{2}(1 + h\mu - h^2\nu)\,a_{320}^2$
$(35)_1$	$a_{321},\ a_{210},\ (1 + l\mu)\,a_{310} - l\nu a_{311} - \frac{1}{2}m\nu a_{321}^2$
$(35)_2$	$a_{321},\ a_{211},\ \mu a_{310} - \nu a_{311} + \frac{1}{2}m\nu a_{321}^2$
$\{\alpha_{32} = 0\}$	$a_{210},\ a_{211},\ a_{310},\ a_{311},$

where for the second group occur the abbreviations

$$u = \tfrac{1}{2}g\,(m - g - l\mu) - \tfrac{1}{2}l\,(h - \nu l),\quad v = m\nu l - hg - hl\mu,$$
$$w = \tfrac{1}{2}m\nu\,(m - g - l\mu) - \tfrac{1}{2}\,(h - \nu l)(h + m\mu).$$

Subgroups of order a power of p in $SLH(4, p^2)$.

20. Let the $GF[p^2]$ be defined by the irreducible congruence $\rho^2 \equiv \mu\rho + \nu \pmod{p}$. The commutator of $[\alpha]$ and $[\beta]$ is $[\delta]$, where $\delta_{i\,i-1} = 0$,

$$\delta_{31} = \beta_{32}\alpha_{21} - \alpha_{32}\beta_{21},\ \delta_{42} = \beta_{43}\alpha_{32} - \alpha_{43}\beta_{32},$$
$$\delta_{41} = \beta_{43}\alpha_{31} - \alpha_{43}\beta_{31} + \beta_{42}\alpha_{21} - \alpha_{42}\beta_{21} - \delta_{42}\,(\alpha_{21} + \beta_{21}).$$

Set $\alpha_{ij} = a_{ij} + \rho A_{ij}$, $\delta_{ij} = d_{ij} + \rho D_{ij}$. By §5, any subgroup of order p^{11} of $G_{p^{12}}$ is defined by a relation

$$\sum_{i=2}^{4} c_i a_{i\,i-1} + \sum_{i=2}^{4} k_i A_{i\,i-1} \equiv 0 \pmod{p}. \tag{37}$$

The commutator subgroup K of $G_{p^{11}}$ contains $B_{41\delta}$, δ arbitrary (§7). If c_3 and k_3 are not both zero, we can take $\alpha_{33} \neq 0, \alpha_{31} = \alpha_{43} = 0, \beta_{31}$ and β_{43} arbitrary, so that δ_{31} and δ_{42} can be made arbitrary. To accomplish the latter when $c_3 = k_3 = 0$, we can take $\alpha_{43} = 0$, $\alpha_{31} \neq 0$, $\beta_{43} \neq 0$, α_{33} and β_{33} arbitrary; the determinant of the coefficients of the latter in δ_{31} and δ_{42} is then $\alpha_{31}\beta_{43} \neq 0$. Hence K is of order p^6. The $(p^5 - 1)/(p - 1)$ subgroups of order p^{10} of $G_{p^{11}}$ are obtained by annexing an independent linear relation

$$\sum_{i=2}^{4} d_i a_{i\,i-1} + \sum_{i=2}^{4} l_i A_{i\,i-1} \equiv 0 \pmod p. \tag{38}$$

21. Consider the commutator subgroup K of $G_{p^{10}}$ defined by (37) and (38). We assume that these are not equivalent to $a_{ss-1} = 0$, the contrary case having been treated in §6. Then, by §7, K contains $B_{41\delta}$, δ arbitrary in the $GF[p^2]$. For brevity we write $(b_{32}a_{21})$ for $b_{32}a_{21} - a_{32}b_{21}$, etc. Then

$$d_{31} = (b_{32}a_{21}) + \nu(B_{32}A_{21}),\ D_{31} = (b_{32}A_{21}) + (B_{32}a_{21}) + \mu\,(B_{32}A_{21}),$$
$$d_{42} = (b_{43}a_{32}) + \nu\,(B_{43}A_{32}),\ D_{42} = (b_{43}A_{32}) + (B_{43}a_{32}) + \mu\,(B_{43}A_{32}).$$

Let first $c_2 = k_2 = d_2 = l_2 = 0$. If $c_3l_3 - d_3k_3 \neq 0$, the group is

$$\{a_{32} = ra_{43} + sA_{43},\ A_{32} = ta_{43} + uA_{43}\}, \tag{39}$$

r, s, t, u, not all zero. If $c_3l_3 - d_3k_3 = 0$, we obtain one of the following:

$$\{A_{43} = ra_{43},\ A_{32} = sa_{32} + ta_{43}\},\ \{A_{43} = ra_{43},\ a_{32} = ta_{43}\}, \tag{40}$$
$$\{a_{43} = 0,\quad A_{32} = sa_{32} + tA_{43}\},\ \{a_{43} = 0,\quad a_{32} = tA_{43}\}. \tag{41}$$

For (39) we have

$$d_{31} = r\,(b_{43}a_{21}) + s\,(B_{43}a_{21}) + \nu t\,(b_{43}A_{21}) + \nu u\,(B_{43}A_{21}),$$
$$D_{31} = t\,(b_{43}a_{21}) + u\,(B_{43}a_{21}) + (r + \mu t)(b_{43}A_{21}) + (s + \mu u)(B_{43}A_{21}),$$
$$d_{42} = (s - \nu t)(b_{43}A_{43}),\ D_{42} = (u - r - \mu t)(b_{43}A_{43}).$$

We may give to $d_{31}, D_{31}, d_{42}, D_{42}$, any values such that

$$(u - r - \mu t)\,d_{42} = (s - \nu t)\,D_{42}. \tag{39$'$}$$

Indeed, if both of the determinants

$$\begin{vmatrix} r & \nu t \\ t & r + \mu t \end{vmatrix},\qquad \begin{vmatrix} s & \nu u \\ u & s + \mu u \end{vmatrix},$$

vanish, then from the irreducibility of $\rho^2-\mu\rho-\nu$ would r, t, s, u, all vanish. If the coefficients in the $(39)'$ are zero, (39) becomes

$$\{a_{32}=\tau a_{43}\}, \qquad \tau=u-\mu t+\rho t. \tag{42}$$

According as (39) *is or is not of the form* (42), *its commutator subgroup is of order p^4 or p^5.* For each of the groups (40), (41), the discussion is similar but simpler, so that only the results need be given. *The commutator subgroup of each of the groups* (40), (41) *is of order p^5*, the d_{ij} and D_{ij} having any values subject to the respective conditions

$$(s+r+\mu rs)\,d_{42}=(1+\nu rs)\,D_{42}, \quad (1+\mu r)\,d_{42}=\nu rD_{42}, \tag{40$'$}$$
$$(1+\mu s)\,d_{42}=\nu sD_{42}, \qquad \mu d_{42}=\nu D_{42}. \tag{41$'$}$$

Let next $c_2l_2-d_2k_2\neq 0$. Then (37) and (38) may be written

$$a_{21}=ra_{43}+sA_{43}+ja_{32}+kA_{32}, \quad A_{21}=ta_{43}+wA_{43}+la_{32}+mA_{32}.$$

We obtain the following values of the d_{ij}, D_{ij}:

	$(b_{32}a_{43})$	$(b_{32}A_{43})$	$(b_{32}A_{32})$	$(B_{32}a_{43})$	$(B_{32}A_{43})$
$d_{31}=$	r	s	$k-\nu l$	νt	νw
$D_{31}=$	t	w	$m-j-\mu l$	$r+\mu t$	$s+\mu w$
$d_{42}=$	-1	0	0	0	$-\nu$
$D_{42}=$	0	-1	0	-1	$-\mu$

It suffices to discuss these equations with $(b_{32}a_{43}), \ldots,$ as variables; for, taking $(b_{32}A_{32})=1$, we may determine a_{43} and b_{43} from $(b_{32}a_{43})$ and $(B_{32}a_{43})$, A_{43} and B_{43} from $(b_{32}A_{43})$ and $(B_{32}A_{43})$. Hence K will be of order p^6 unless every determinant of the fourth order in the above matrix is zero. That of the 1st, 2nd, 4th, 5th columns equals

$$-(\nu t-s)^2-\mu(\nu t-s)(r-w+\mu t)+\nu(r-w+\mu t)^2.$$

In view of the irreducibility of (4), this vanishes only if

$$s=\nu t,\; r=w-\mu t. \tag{43}$$

When (43) holds every determinant of order 4 vanishes, and

$$\begin{aligned} d_{31}(m-j-\mu l)+D_{31}(\nu l-k)+d_{42}\,&[(w-\mu t)(m-j-\mu l)-t(k-\nu l)] \\ &+D_{42}\,[\nu t(m-j-\mu l)-w(k-\nu l)]=0. \end{aligned} \tag{44}$$

If also
$$m-j-\mu l=0,\; k-\nu l=0, \tag{45}$$

then all determinants of order 3 in the matrix vanish identically, and we obtain

$$D_{31} + td_{42} + wD_{42} = 0, \; d_{31} + (w - \mu t)\, d_{42} + vtD_{42} = 0. \tag{46}$$

When conditions (43) *and* (45) *all hold, the group becomes*

$$\{a_{21} = (r + \rho t)\, a_{43} + (j + \rho l)\, a_{32}\}, \tag{47}$$

and its commutator subgroup K is of order p^4. When (43) *but not* (45) *hold,** *K is of order p^5. When* (43) *do not hold, K is of order p^6.*

Let finally $c_2 l_2 - d_2 k_2 = 0$, but c_2, l_2, d_2, k_2, not all zero. Then (37) and (38) may be written $ca_{21} + kA_{21} + f_1 \equiv 0, f_2 \equiv 0$, where f_1 and f_2 are linear functions of $a_{32}, a_{43}, A_{32}, A_{43}$, and c, k are not both zero. *If the coefficient of either A_{43} or a_{43} in f_2 is not zero, I find that the commutator subgroup K is of order p^6*, each $\delta_{ij}\,(j < i - 1)$ being arbitrary. There remain the groups

$$\{A_{32} = la_{32}, a_{21} = \phi\}, \; \{A_{32} = la_{32}, A_{21} = ca_{21} + \phi\}, \; \phi \equiv rA_{43} + sa_{43} + ka_{32}, \tag{48}$$

$$\{a_{32} = 0, \quad a_{21} = \psi\}, \; \{a_{32} = 0, \quad A_{21} = ca_{21} + \psi\}, \; \psi \equiv rA_{43} + sa_{43} + kA_{32}. \tag{49}$$

For each of these groups (48), (49), *K is of order p^5*, the d_{ij}, D_{ij} being subject to a single condition:

$$(1 + \mu l)\, d_{31} - vlD_{31} + (r - vls)\, D_{42} + (s + \mu ls - lr)\, d_{42} \equiv 0, \tag*{(48)$'_1$}$$

$$(1 + vlc)\, D_{31} - (c + l + \mu lc)\, d_{31} + (r - vls)\, D_{42} + (s + \mu ls - lr)\, d_{42} \equiv 0, \tag*{(48)$'_2$}$$

$$vD_{31} - \mu d_{31} + vsD_{42} + (r - \mu s)\, d_{42} \equiv 0, \tag*{(49)$'_1$}$$

$$(1 + \mu c)\, d_{31} - vcD_{31} + vsD_{42} + (r - \mu s)\, d_{42} \equiv 0. \tag*{(49)$'_2$}$$

The largest m-ary linear group containing self-conjugately a given subgroup of order a power of p.

22. *The $p^{\mu n}(p^n - 1)^m$ operators transforming $G_{p^{\mu n}}$ into itself are*

$$(\delta_{ij}), \; \delta_{ij} = 0\,(j > i), \; \delta_{ii} \neq 0. \tag{50}$$

Let (δ_{ij}) be a general matrix. Equating the coefficients of ξ_j in the functions by which $[A_{ij}](\delta_{ij})$ and $(\delta_{ij})[a_{ij}]$ replace ξ_i, we get

$$\sum_{k=j+1}^{m} \delta_{ik} A_{kj} = \sum_{l=1}^{i-1} a_{il}\, \delta_{lj} \; (i, j = 1, \ldots, m). \tag{51}$$

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* Then $a_{21} = (r + \rho t)\, a_{43} + xa_{32} + yA_{32}$, $y \neq \rho x$.

The A_{ij} $(j < i)$ may be taken arbitrary, while the α_{ij} are then to be determined. For $i = j = 1$, (51) gives $\delta_{1k} = 0$ $(k = 2, \ldots, m)$. To proceed by induction, suppose that $\delta_{ik} = 0$ $(i = 1, \ldots, r;\ k = i + 1, \ldots, m)$. Then (51) for $i = j = r + 1$ becomes

$$\sum_{k=r+2}^{m} \delta_{r+1k} A_{kr+1} = 0,$$

whence $\delta_{r+1k} = 0$ $(k = r + 2, \ldots, m)$. Hence (δ_{ij}) is of the form (50). For $j \geqq i$, conditions (51) are now identities; for $j \leqq i - 1$, (51) becomes

$$\sum_{k=j+1}^{i} \delta_{ik} A_{kj} = \sum_{l=j}^{i-1} \alpha_{il} \delta_{lj} \qquad (j \leqq i - 1). \qquad (52)$$

Since $\delta_{jj} \neq 0$, (52) serves to express α_{ij} in terms of α_{il} $(l = j + 1, \ldots, i - 1)$, A's and δ's and hence ultimately in terms of the A's and δ's.

23. The $p^{\mu n}(p^n - 1)^{m-1}(p^{2n} - 1)$ *operators transforming* $\{\alpha_{ss-1} = 0\}$ *into itself are*

$$(\delta_{ij}),\ \delta_{ij} = 0\,(j > i) \text{ except } \delta_{s-1s},\ \delta_{ii} \neq 0\,(i = 1, \ldots, m;\ i \neq s, s - 1), \qquad (53)$$
$$\Delta_{s-1s} \equiv \delta_{s-1s-1}\delta_{ss} - \delta_{s-1s}\delta_{ss-1} \neq 0.$$

For $i = j = 1, 2, \ldots, s - 2$ in (51), we get $\delta_{ij} = 0$ $(i = 1, \ldots, s - 2;$ $j = i + 1, \ldots, m)$ as in §22. For $i = j = s - 1$, (51) becomes $\sum_{k=s+1}^{m} \delta_{s-1k} A_{ks-1} = 0$, whence $\delta_{s-1k} = 0$, $k \geqq s + 1$. For $i = j = s$, (51) gives $\delta_{sk} = 0$, $k \geqq s + 1$. By induction we prove that $\delta_{ij} = 0, j > i, i \geqq s$. Hence (δ_{ij}) is of the form (53). Conditions (51) are now identities for $j \geqq i$. Let next $j \leqq i - 1$. For $j \neq s$, $s - 1$, (51) serves to express α_{ij} in terms of the α_{il} $(l = j + 1, \ldots, i - 1)$, A's and δ's, since the coefficient of α_{ij} is $\delta_{jj} \neq 0$, while that of α_{il} $(l < j)$ is zero. The two conditions (51) given by $j = s$, $j = s - 1$, in which we may assume that $i > s$, serve to express α_{is-1} and α_{is} in terms of the α_{il} $(l = s + 1, \ldots, i - 1)$, since $\Delta_{s-1s} \neq 0$.

24. *If the non-vanishing* c_i *are* $c_{i_1}, \ldots, c_{i_v}$, $v \geqq 2$, the $p^{\mu n}(p^n - 1)^{m-v+1}$ *operators transforming into itself* $\left\{\sum_{i=2}^{m} c_i \alpha_{ii-1} = 0\right\}$ *are*

$$(\delta_{ij}),\ \delta_{ij} = 0\,(j > i),\ \delta_{ii} \neq 0,\ \delta_{ii}\delta_{i-1i-1}^{-1}\,(i = i_1, \ldots, i_v) \text{ all equal.} \qquad (54)$$

... A_{u-1} may be taken arbitrary. ... while the remaining conditions reduce ... of the A's and δ's. In particular, for

$$\ldots = \ldots \delta_{i-u-1} \qquad (i = 2, \ldots, m).$$

$$\ldots = \sum_{i=2}^{m} c_i A_{u-1} \delta_{u} \delta_{i-1u-1}^{-1} = 0$$

... = 1, giving the final conditions (54).

... $= \ldots \neq s, j \neq s$), group (25) takes the form

$$\ldots = 0, \sum_{\substack{i=2,\ldots,m \\ i \neq s}} \gamma_i a_{u-1} = 0 \Big\}, \qquad (\text{not every } \gamma_i = 0), \qquad (55)$$

... considered in §§26, 27. Suppose here that for each integer s, ... $\neq 0\,(i \neq s, j \neq s)$. Then for any given s we may take ... and determine the remaining a_{u-1} to satisfy the conditions on (25). ... in §23, an operator (δ_{ij}) commutative with (25) has every ... As in §24, the equations

$$\sum_{i=2}^{m} c_i A_{u-1} \delta_{u} \delta_{i-1u-1}^{-1} = 0, \sum_{i=2}^{m} k_i A_{u-1} \delta_{u} \delta_{i-1u-1}^{-1} = 0$$

must follow from

$$\sum_{i=2}^{m} c_i A_{u-1} = 0, \sum_{i=2}^{m} k_i A_{u-1} = 0.$$

Let ... $\neq 0$, so that $A_{u-1}(i = 2, \ldots, m; i \neq r, t)$ may be given arbitrary values. Equating the values of A_{rr-1} and A_{tt-1} obtained by solving the two pairs of equations, we get

$$\frac{\delta_{r-1}}{\delta_{rr}} \begin{vmatrix} c_i \delta_{u} \delta_{i-1u-1}^{-1} & c_t \\ k_i \delta_{u} \delta_{i-1u-1}^{-1} & k_t \end{vmatrix} = \begin{vmatrix} c_i c_t \\ k_i k_t \end{vmatrix}, \quad \frac{\delta_{t-1t-1}}{\delta_{tt}} \begin{vmatrix} c_r & c_i \delta_{u} \delta_{i-1u-1}^{-1} \\ k_r & k_i \delta_{u} \delta_{i-1u-1}^{-1} \end{vmatrix} = \begin{vmatrix} c_r c_i \\ k_r k_i \end{vmatrix}$$

for $i = 2, \ldots, m; i \neq r, t$. Hence must

$$D_{it}(\delta_{rr}\delta_{i-1i-1} - \delta_{ii}\delta_{r-1r-1}) = 0, \; D_{ir}(\delta_{tt}\delta_{i-1i-1} - \delta_{ii}\delta_{t-1t-1}) = 0.$$

If $D_{it} = D_{ir} = 0$, then $c_i = k_i = 0$ and the preceding are identities.

THEOREM. *For each integer* s, $2 \overline{\gtrless} s \overline{\gtrless} m$, *let at least one of the* $D_{ij} \neq 0$ $(i \neq s, j \neq s)$. *Let, in particular,* $D_{rt} \neq 0$. *Denote by* $i_1, \ldots, i_v$ *the integers* i *such that* $2 \overline{\gtrless} i \overline{\gtrless} m$, $i \neq r$, $i \neq t$, *and such that* c_i, k_i *are not both zero. If there occurs among the* $i_1, \ldots i_v$ *an integer* i *for which* $D_{it} \neq 0$, $D_{ir} \neq 0$, *then the* $p^{\mu n}(p^n - 1)^{m-v-1}$ *operators transforming* (25) *into itself are the* (δ_{ij}),

$$\delta_{ij} = 0\,(j > i),\ \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{rr}}{\delta_{r-1r-1}} = \frac{\delta_{tt}}{\delta_{t-1t-1}}\ (i = i_1, \ldots, i_v). \qquad (56)$$

In the contrary case, let $i_1, \ldots, i_w$ *be the integers* i *for which* $D_{it} \neq 0$, $D_{ir} = 0$, *and* $i_{w+1}, \ldots, i_v$ *the remaining* i *for which therefore* $D_{it} = 0$, $D_{ir} \neq 0$; *then the* $p^{\mu n}(p^n - 1)^{m-v}$ *operators transforming* (25) *into itself are the* (δ_{ij}),

$$\delta_{ij} = 0\,(j > i),\ \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{rr}}{\delta_{r-1r-1}}\ (i = i_1, \ldots, i_w),$$

$$\frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{tt}}{\delta_{t-1t-1}}\ (i = i_{w+1}, \ldots, i_v). \qquad (57)$$

26. Consider group (55) with at least two γ_i not zero. Any particular $A_{ii-1}\,(i \neq s)$ can be taken arbitrarily. As in §23, every $\delta_{ij} = 0\,(j > i)$ except possibly δ_{s-1s}, while those conditions (51) which do not now reduce to identities serve to express the $\alpha_{ij}\,(j \overline{\gtrless} i - 1)$ in terms of the A's and δ's. In particular, for $j = i - 1$, $i \neq s$, $i \neq s + 1$, (51) gives

$$\alpha_{ii-1} = A_{ii-1}\delta_{ii}\delta^{-1}_{i-1i-1}\,(i \neq s - 1, s, s + 1),$$

$$\alpha_{s-1s-2} = \delta^{-1}_{s-2s-2}\,(\delta_{s-1s-1}A_{s-1s-2} + \delta_{s-1s}A_{ss-2});$$

while for $i = s + 1, j = s - 1$, and for $i = s + 1, j = s$, (51) gives

$$\delta_{s+1s+1}A_{s+1j} = \alpha_{s+1s}\delta_{sj} + \alpha_{s+1s-1}\delta_{s-1j}\,(j = s - 1, s),$$

which determine $\alpha_{s+1s}, \alpha_{s+1s-1}$. The equation obtained upon substituting these values of the α_{ii-1} in $\Sigma\gamma_i\alpha_{ii-1} = 0$ must follow from $\Sigma\gamma_i A_{ii-1} = 0$. The coefficients of A_{ss-2} and A_{s+1s-1} must vanish, whence

$$\gamma_{s-1}\delta_{s-1s} = 0,\ \gamma_{s+1}\delta_{s-1s} = 0.$$

Further, for the non-vanishing γ_i, there must result equal values of

$$\gamma_i\delta_{ii}\delta^{-1}_{i-1i-1} : \gamma_i,\quad \gamma_{s+1}\delta_{s+1s+1}\delta_{s-1s-1}\Delta^{-1}_{s-1s} : \gamma_{s+1}\,(i \neq s, s + 1).$$

THEOREM. *Let the non-vanishing* γ_i *be* $\gamma_{i_1}, \ldots, \gamma_{i_v}, v \gtreqless 2$. *If either* $\gamma_{s+1} \neq 0$ *or* $\gamma_{s-1} \neq 0$, *the* $p^{\mu n}(p^n - 1)^{m-v+1}$ *operators transforming* (55) *into itself are given by* (54). *If* $\gamma_{s+1} = \gamma_{s-1} = 0$, *the* $p^{\mu n}(p^n - 1)^{m-v}(p^{2n} - 1)$ *operators are given by* (53), *subject to the further condition that the* $\delta_{ii}\delta_{i-1i-1}^{-1}$ $(i = i_1, \ldots, i_v)$ *shall be equal.*

27. Consider group (55) with $\gamma_i = 0\ (i \neq r)$, viz. $\{\alpha_{ss-1} = 0,\ \alpha_{rr-1} = 0\}$. We may take $r > s$. As in §23,

$$\delta_{ij} = 0\ (i = 1, \ldots, s-2;\ j > i),\ \delta_{s-1k} = 0\ (k = \gtreqless s+1),\ \sum_{k=s+1}^{m} \delta_{sk}A_{ks} = 0.$$

If $r = s+1$, the latter gives merely $\delta_{sk} = 0\ (k \gtreqless s+2)$. Then (51) for $i = j = s+1$ gives $\delta_{s+1k} = 0\ (k \gtreqless s+2)$. By induction, $\delta_{ik} = 0$, $(i = s+1, \ldots, m;\ k > i)$. Hence every $\delta_{ij} = 0\ (j > i)$ except δ_{s-1s} and $\delta_{ss+1} \equiv \delta_{r-1r}$. Similarly, if $r > s+1$, we get $\delta_{ij} = 0\ (j > i)$ except $\delta_{s-1s}, \delta_{r-1r}$.

Conditions (51) are now identities if $j \lessgtr i$. Let next $j \lessgtr i - 1$.

Let first $r > s+1$. Then $\delta_{ii} \neq 0\ (i \neq s-1, s, r-1, r)$, $\Delta_{s-1s} \neq 0$, $\Delta_{r-1r} \neq 0$. For $j \neq s-1, s, r-1, r$, (51) serves to express α_{ij} in terms of the $\alpha_{il}\ (l = j+1, \ldots, i-1)$, A's and δ's. Proceeding as at the end of §23, we conclude that conditions (51) for $j \lessgtr i-1$ merely serve to express the α_{ij} in terms of the A's and δ's.

For $r = s+1$, we have $\delta_{ii} \neq 0\ (i \neq s-1, s, s+1)$, and

$$\Delta \equiv \begin{vmatrix} \delta_{s-1s-1} & \delta_{s-1s} & 0 \\ \delta_{ss-1} & \delta_{ss} & \delta_{ss+1} \\ \delta_{s+1s-1} & \delta_{s+1s} & \delta_{s+1s+1} \end{vmatrix} \neq 0.$$

For $j \neq s-1, s+1$, (51) serves to express α_{ij} in terms of the $\alpha_{il}(l = j+1, \ldots, i-1)$, A's, and δ's. For $j = s-1, s, s+1$, with $i > s+1$, (51) determines α_{is-1}, α_{is}, α_{is+1}, the determinant of their coefficients being Δ. There remain the cases $i = s+1, j = s;\ i = s+1, j = s-1;\ i = s, j = s-1$, for which (51) becomes, respectively,

$$0 = \alpha_{s+1s-1}\delta_{s-1s},\ \delta_{s+1s+1}A_{s+1s-1} = \alpha_{s+1s-1}\delta_{s-1s-1},\ \delta_{ss+1}A_{s+1s-1} = 0.$$

Hence $\delta_{s-1s} = \delta_{ss+1} = 0$, whence every $\delta_{ii} \neq 0$. The second condition thus determines α_{s+1s-1}. Hence the $\alpha_{ij}\ (j \lessgtr i-1)$ are determined in terms of the A's, δ's.

THEOREM. *The $p^{\mu n}(p^n-1)^m$ operators transforming $\{\alpha_{ss-1}=\alpha_{s+1s}=0\}$ into itself are given by* (50). *If $r>s+1$, the $p^{\mu n}(p^n-1)^{m-3}(p^{2n}-1)^2$ operators transforming $\{\alpha_{ss-1}=0,\ \alpha_{rr-1}=0\}$ into itself are the (δ_{ij}),*

$$\delta_{ij}=0\ (j>i)\ \textit{except}\ \delta_{s-1s}\ \textit{and}\ \delta_{r-1r},\ \Delta_{s-1s}\neq 0,$$
$$\Delta_{r-1r}\neq 0,\ \delta_{ii}\neq 0\ (i\neq s-1,\ s,\ r-1,\ r). \quad (58)$$

28. For the group (28), we proceed as in §23 except for the step $i=j=s-1$, instead of which we employ (51) for $i=s-1, j=s$, and get $\delta_{s-1k}=0\ (k\overline{>}s+1)$. The values of the α_{ii-1} and α_{s+1s-1} are the same as in §26. Substituting them in $\alpha_{s+1s-1}=\alpha_{s-1s-2}+\Sigma x_i\alpha_{ii-1}$, replacing A_{s+1s-1} by $A_{s-1s-2}+\Sigma x_iA_{ii-1}$, and equating the coefficients of each A_{ii-1} and of A_{ss-2}, we get

$$\delta_{s-1s}=\delta_{ss-1}=0,\ x_i\delta_{s+1s+1}\delta_{ss}\Delta^{-1}_{s-1s}=x_i\delta_{ii}\delta^{-1}_{i-1i-1}(i\neq s,\ s+1).$$

THEOREM. *If the non-vanishing x_i are $x_{i_1},\ldots,x_{i_v}$, the $p^{n(\mu-1)}(p^n-1)^{m-v}$ operators transforming* (28) *into itself are the (δ_{ij}),*

$$\delta_{ij}=0(j>i),\ \delta_{ss-1}=0,\ \delta_{ii}\delta^{-1}_{i-1i-1}=\delta_{s+1s+1}\delta^{-1}_{s-1s-1}\ (i=i_1,\ldots,i_v). \quad (59)$$

29. Consider group (21) with $k_1=1, k_{s+1}=0$ (§13, end). We can take every A arbitrary except A_{ss-1} and A_{s+1s-1}. By (51) for $i=j=1,\ldots,s-2$, we get $\delta_{ij}=0\ (i=1,\ldots,s-2; j>i)$. For $i=s-1, j=s$, (51) gives $\delta_{s-1k}=0\ (k\overline{>}s+1)$. Then for $i=j=s-1$, (51) becomes $\delta_{s-1s}A_{ss-1}=0$. But we may take $A_{ss-1}\neq 0$. Hence $\delta_{s-1k}=0\ (k\overline{>}s)$. Then (51) for $i=j=s,\ldots,m$ gives $\delta_{ik}=0\ (i\overline{>}s, k>i)$. Now (52), for $j=i-1, j=i-2$, give

$$\alpha_{ii-1}=\frac{A_{ii-1}\delta_{ii}}{\delta_{i-1i-1}},\ \alpha_{ii-2}=\frac{\delta_{ii-1}A_{i-1i-2}+\delta_{ii}A_{ii-2}}{\delta_{i-2i-2}}-\frac{A_{ii-1}\delta_{ii}\delta_{i-1i-2}}{\delta_{i-1i-1}\delta_{i-2i-2}}. \quad (60)$$

Substituting these in the two equations (21), eliminating A_{s+1s-1} and then A_{s+1s} by (21) written in the A's, we get

$$\frac{\delta_{ss}}{\delta_{s-1s-1}}=\frac{\delta_{s+1s+1}}{\delta_{ss}},\ \sum_{\substack{i=2,\ldots,m\\ i\neq s,s+1}}k_iA_{ii-1}\Big(\frac{\delta_{ii}}{\delta_{i-1i-1}}-\frac{\delta_{s+1s+1}}{\delta_{s-1s-1}}\Big)-\frac{A_{ss-1}\beta}{\delta_{ss}\delta_{s-1s-1}}=0,$$

where β is given below. This must be an identity in the A's.

THEOREM. *Denote by* Γ *the group* (21) *with* $k_1 = 1, k_{s+1} = 0$. *Let the non-vanishing* k_i *be* $k_{i_1}, \ldots, k_{i_v}$. *The* $p^{n(\mu-1)}(p^n - 1)^{m-v-1}$ *operators commutative with* Γ *are the* (δ_{ij}),

$$\delta_{ij} = 0\ (j > i),\ \frac{\delta_{ss}}{\delta_{s-1\,s-1}} = \frac{\delta_{s+1\,s+1}}{\delta_{ss}},\ \frac{\delta_{ii}}{\delta_{i-1\,i-1}} = \frac{\delta_{s+1\,s+1}}{\delta_{s-1\,s-1}}\ (i = i_1, \ldots, i_v),$$
$$\beta \equiv c_s\delta_{s\,s-1}\delta_{s+1\,s+1} + c_{s+1}\delta_{s+1\,s}\delta_{ss} = 0. \tag{61}$$

30. THEOREM. *Denote by* H *the group* (23) *with* $k_1 = 1$, $k_{s+1} = 0$. *Let the non-vanishing* k_i *be* $k_{i_1}, \ldots, k_{i_v}$. *The operators commutative with* H *are*

$$(\delta_{ij}),\ \delta_{ij} = 0\ (j > i),\ \frac{\delta_{ii}}{\delta_{i-1\,i-1}} = \frac{\delta_{s+1\,s+1}}{\delta_{s-1\,s-1}} = \frac{\delta_{s+2\,s+2}}{\delta_{ss}} (i = i_1, \ldots, i_v), \tag{62}$$

$$C \equiv c_s\delta_{s\,s-1}\delta_{s+1\,s+1} + c_{s+1}\delta_{s+1\,s}\delta_{ss} + c_{s+2}\delta_{s+2\,s+1}\delta_{s-1\,s-1} = 0,$$

with in case $c_{s+1} \neq 0$, *the further condition* $\delta_{ss}^2 = \delta_{s+1\,s+1}\delta_{s-1\,s-1}$.

The proof proceeds as in §29. We substitute the values (60) in the two equations (23), eliminate $A_{s+1\,s-1}$ and then $A_{s\,s-1}$ by (23) written in the A's. From $\Sigma c_i a_{i\,i-1} = 0$ follows, as in §24, that the $\delta_{ii}\delta_{i-1\,i-1}^{-1} (i = i_1, \ldots, i_v)$ are equal. From $f_a + \Sigma k_i a_{i\,i-i} = 0$, we get

$$\left(\frac{\delta_{s+1\,s+1}}{\delta_{s-1\,s-1}} - \frac{\delta_{s+2\,s+2}}{\delta_{ss}}\right)\left(c_{s+2}A_{s+2\,s} - \tfrac{1}{2}c_{s+2}A_{s+2\,s+1}A_{s+1\,s} - \frac{c_{s+2}A_{s+2\,s+1}\delta_{s+1\,s}}{\delta_{s+1\,s+1}}\right)$$
$$+ \sum_{\substack{i=2,\ldots,m\\ i\neq s,\,s+1}} k_i A_{i\,i-1}\left(\frac{\delta_{ii}}{\delta_{i-1\,i-1}} - \frac{\delta_{s+1\,s+1}}{\delta_{s-1\,s-1}}\right) - \frac{CA_{s+1\,s}}{\delta_{ss}\delta_{s-1\,s-1}} = 0.$$

CHICAGO, *November* 2, 1904.

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*Concerning Certain 4-Space Quintic Configurations of Point Ranges and Congruences, and Their Sphere Analogues in Ordinary Space.**

BY C. J. KEYSER.

I. *Introductory Considerations.*

Meaning of terms. Consistently with the Plücker nomenclature as extended by Lie † and others, I shall employ the term sphere complex to denote the assemblage of spheres (of ordinary space) that are all of them orthogonal to a same sphere; the term sphere congruence to signify the intersection of two sphere complexes, i.e. the totality of spheres common to them; and the term sphere range to mean the common intersection of three independent sphere complexes. Analogously the term point complex, or lineoid, will denote the assemblage of points of an ordinary (linear) 3-space of point 4-space, while the intersections of two, and of three independent, lineoids will be called, respectively, congruence and range of points, the latter term, point range, being thus employed in its usual sense.

Fundamental correlations. Ordinary space is 4-dimensional in sphere complexes as well as in spheres. The sphere and the sphere complex are so related that, if either be chosen as element, the other is thereby determined as reciprocal element. It is accordingly evident that by employing the sphere complex as element, a theory of ordinary space may be constructed which will be geometrically precisely parallel to "sphere geometry" and algebraically identical with it. These dual theories may be conveniently denoted, the latter by ${}_4T_s$, and the former by ${}_4T_{sc}$. Like statements, it is well known, are valid in case of point 4-space, the point and the lineoid (point complex) being taken as (dual) elements. The theories

* Read under a slightly different title before the American Mathematical Society, April 30, 1904.

† Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen. Math. Ann., Vol. V.

of this space which depend primarily upon these elements will be denoted, respectively, by the symbols ${}_4T_p$ and ${}_4T_{pc}$. Again, ordinary space is 6-dimensional in sphere ranges and in sphere congruences, as is 4-space in ranges and in congruences of points. The 3-space theories of the sphere range and the sphere congruence will be denoted by ${}_6T_{sr}$ and ${}_6T_{sc}$, and the 4-space theories of the point range and the point congruence by ${}_6T_{pr}$ and ${}_6T_{pc}$. Suppose established a unique and reciprocal correspondence: (1) in 3-space, between spheres and sphere complexes, and therewith between sphere ranges and sphere congruences; (2) in 4-space, between points and lineoids, and therewith between point ranges and point congruences; (3) between the spheres of 3-space and the points of 4-space, and therewith, respectively, between the sphere complexes, sphere congruences, and sphere ranges, of 3-space, and the lineoids, the point congruences, and the point ranges, of 4-space. This done, it is obvious that a fact-to-fact correlation will subsist: (1) between the theories ${}_4T_s$ and ${}_4T_p$, and between their respective reciprocals ${}_4T_{sc}$ and ${}_4T_{pc}$; (2) between the theories ${}_6T_{sr}$ and ${}_6T_{pr}$, and between their reciprocals ${}_6T_{sc}$ and ${}_6T_{pc}$.

Aim and method. While the mentioned 3-space doctrines are analytically and logically identical, each to each, with their 4-space correlates, the elements of the latter—either in themselves or owing, it may be, to a biological accident—are intuitionally far simpler and more tractable than the former, lending themselves much more readily to both interrogation and answer. It so becomes, strangely enough, a not unimportant principle of economy to construct the 4-space theories first, and then, using them as source and type, to derive their 3-space correspondents by the simple process of translation, or substitution of notions. It is the object of this note to illustrate the advantage of this order of procedure by following it in the establishment of several propositions, themselves of no little interest and importance in the theories concerned.

II. *Demonstration of Theorems.*

Preliminary propositions. Let r_1, r_2, r_3 denote three independent point ranges of 4-space. Two of them, as r_2 and r_3, being equivalent to four independent points, determine* a lineoid L. L and r_1 have in common one and but one point P. P and r_3 (say) determine in L a point congruence C, and this contains

* Such simple propositions in ${}_4T_p$ and ${}_4T_{pc}$ are assumed.

one and but one point Q of r_2. The range r_4 determined by P and Q, and no other range, has a point in common with each of the given ranges. We have, therefore, in ${}_6T_{pr}$,

THEOREM (a).—*There is one and but one point range having a point in common with each of three given independent point ranges.*

Other aspects of the matter are worthy of indication. Thus the fourth range is the common intersection of the three lineoids determined by the three given ranges taken in pairs. Again, it is the common intersection of the three point congruences determined by the three mentioned lineoids taken in pairs. Once more, a point range, being common to ∞^2 point congruences, may be regarded as the envelope of a *hyperpencil* of point congruences; whence it is seen that, given three independent hyperpencils of point congruences, there is one and but one other such hyperpencil having a congruence in common with each of the given hyperpencils.

On replacing the notions, point range, point congruence, lineoid, and point, by their respective reciprocals, there results immediately, in ${}_6T_{pc}$,

THEOREM (a).—*There is one and but one point congruence collineoidal with each of three given independent point congruences.*

The fourth congruence is common to, contains, is determined by, the three points determined by the given congruences taken in pairs, and contains the three ranges determined by the same points taken in pairs. Again, a point congruence, as it contains ∞^2 point ranges, may be regarded as their locus, i. e. as a hyperpencil of point ranges; whence it appears that there is one and but one hyperpencil of point ranges having a range in common with each of three given independent hyperpencils of ranges.

Appropriate substitution of ideas at once yields the following correlative (in themselves less easily apprehended) propositions concerning spheres in 3-space, in ${}_6T_{sr}$,

THEOREM (a).—*There is one and but one sphere range having a sphere in common with each of three given independent sphere ranges.*

The fourth sphere range is the assemblage of spheres common to the three sphere complexes determined by the given ranges taken in pairs; it is also com-

mon to the three sphere congruences determined by the same complexes taken in pairs. A sphere range, being common to ∞^2 sphere congruences, may be regarded as their envelope, i.e. as a hyperpencil of sphere congruences; whence it follows, that there is always one and but one hyperpencil of sphere congruences having a congruence in common with each of three given independent hyperpencils of congruences. Reciprocally, in ${}_6T_{sc}$,

THEOREM (a).— *There is one and but one sphere congruence which with each of three given independent sphere congruences determines a sphere complex.*

The fourth congruence is common to, determined by, contains, the three spheres determined by the three given congruences taken in pairs; it contains the three sphere ranges determined by the same three spheres taken in pairs. A sphere congruence, as it contains ∞^2 sphere ranges, may be thought as their locus, i.e. as a hyperpencil of sphere ranges; whence follows the proposition that, given three independent hyperpencils of sphere ranges, there is always one and only one fourth hyperpencil having a range in common with each of the given hyperpencils.

The foregoing preliminary propositions lead directly to the matter proper of this paper.

Quintic configuration of point ranges. Denote by the symbols, aa, ab, ac, ad, four independent point ranges.

By theorem (a), in ${}_6T_{pr}$, the four triplets furnished by the given ranges determine four additional ranges, ae, be, ce, de, which may be associated with their respective determining triplets as in the following scheme:

$$\left.\begin{matrix} ab \\ ac \\ ad \end{matrix}\right\} ae \qquad \left.\begin{matrix} ac \\ ad \\ aa \end{matrix}\right\} be \qquad \left.\begin{matrix} ad \\ aa \\ ab \end{matrix}\right\} ce \qquad \left.\begin{matrix} aa \\ ab \\ ac \end{matrix}\right\} de. \tag{1}$$

The intersections of the four e-ranges with their corresponding triplets furnish 12 points which, taken in pairs as in the scheme below, determine 6 new ranges:

$$\left.\begin{array}{lcccccc} \text{points}: & (aa,\, be), & (aa,\, ce), & (aa,\, de), & (ab,\, ce), & (ab,\, de), & (ac,\, de), \\ & (ab,\, ae), & (ac,\, ae), & (ad,\, ae), & (ac,\, be), & (ad,\, be), & (ad,\, ce); \\ \text{ranges}: & cd\;, & bd\;, & bc\;, & ad\;, & ac\;, & ab\;. \end{array}\right\} \tag{2}$$

It is now to be shown that of these ranges any pair, as ab and cd, involving the four letters a, b, c, d, have a common point. To this end consider the lineoids L and L' determined, respectively, by the range pairs ae, be and ce, de. By inspection of the first two columns of scheme (1), it is seen that ac and ad have each a point in common with ae and with be and so have each two points in L. The ranges ac and ad, therefore, lie in L. Range ab having, as may be seen in the last column of (2), a point in ac and another in ad, it follows that ab is entirely in L. From the first column of (2), it is evident that cd, too, is in L. By reference to the last two columns of (1) and the initial and final columns of (2), it may be readily shown that ab and cd are both of them also in L'. Being in both L and L', the ranges ab and cd are in the congruence (L, L'), and accordingly have a common point (ab, cd). It may be shown analogously, or from symmetry, immediately inferred, that the range pairs ac, bd and ad, bd also determine points (ac, bd) and (ad, bc).

On the other hand, of the six ranges (2) no pair involving but three distinct letters have a common point. For if such a pair, as ab, ac, be supposed to have a common point, it would follow that all the ranges (2) would each have a point in each of the other five ranges, accordingly that the 12 points (2) would all of them belong to a same point congruence and thence that the latter would contain the four ranges originally given—a result incompatible with the hypothesis that these are independent. The ranges of such a pair as ab, ac, therefore, determine a lineoid.

Consider now the lineoids L_1, L_2, L_3 determined, respectively, by the range pairs: ab, ac; ab, ad; ac, ad. From the last two columns of (2), it is seen that L_1 contains two points of ad and two of de, whence it appears from the third column of (2) that bc, as it has a point in ad and another in de, is itself contained in L_1. Accordingly the lineoid determined by two of any three ranges (of the six (2)) involving but three letters, contains the third range. Hence L_2 contains bd; and L_3, cd. It follows that the three points

$$(ab,\ cd),\ (ac,\ bd),\ (ad,\ bc) \tag{3}$$

all lie in each of the lineoids L_1, L_2, L_3. These three L's are independent, for, if not, they contain a common congruence; this last must, then, contain such a range pair, e, g, as ab, ac, a relationship above found to be impossible. The three L's, therefore, determine a point range, and this range—call it ae—contains the three points (3).

In the foregoing argument the four given ranges enter precisely alike. From this symmetry it follows that if ae were collineoidal with two of the given ranges, it would be so with all of them, which has been shown to be impossible. Hence, ae with any three of the given ranges constitute a set of four independent ranges, and may be called the *associate* range of the given set. Suppose a range as ad of the given set replaced by ae. The new set of four determines a fifth associate range ad', say. We now show that ad' and ad coincide. In case of the original set, ae, by theorem (a), of ${}_6T_{pr}$, is uniquely determined by the independents ad, bd and cd; hence, in case of the new set, ad' is uniquely determined by ae, be and ce; but these same three, as appears in scheme (1), also determine ad; ad and ad' are, therefore, identical.

Accordingly we may state, in ${}_6T_{pr}$,

THEOREM (b).—*In point 4-space any four given independent point ranges uniquely determine a fifth associate point range. The five ranges compose a quintic configuration such that any four of the ranges determine the fifth as their associate.*

The operation O_{pr} of determination (construction). The operation O_{pr} of finding the associate of four given independent ranges may, of course, be viewed either as objective or as subjective. Viewing it as subjective, O_{pr} may be analyzed with sufficient minuteness into the following *ordered* acts of attention. It consists, namely, of *regarding*: (*i*) in any order the four ranges determined according to theorem (a), of ${}_6T_{pr}$, by the given set; (*ii*) in any order the 12 points furnished by the given ranges and the four ranges found by (*i*); (*iii*) in any order the 6 ranges determined by the six point pairs afforded as in scheme (2) by the 12 points; (*iv*) in any order the 3 points given by the last 6 ranges properly paired; (*v*) the range (associate) containing the last 3 points.

Applied to a set, as that originally given, of four independent ranges, O_{pr} presents 11 new ranges, which, together with the given set, constitute an interesting configuration, ${}_{15}C_{pr}$, of 15 ranges.

Properties of ${}_{15}C_{pr}$. Among the noteworthy properties of this configuration, the following may be stated at once as being immediately evident or readily demonstrable.

A.—The 15 *ranges determine by intersection* 15 *points such that each point contains* 3 *of the ranges, and each range* 3 *of the points.* The point and the range not being reciprocal elements of 4-space, the 15 ranges are not the totality of ranges

determined by the points, but the ranges of any one of the 15 triplets lie in and determine a lineoid, making 15 lineoids reciprocal to the 15 points.

B.—The 15 *ranges furnish* 30 *sets of* 4 *independent ranges each, these sets being so related that the operation O_{pr}, applied independently to any two of them, presents one and the same configuration ${}_{15}C_{pr}$ in such a way that no range is replaced by another but that the orders of their presentation are in general different.* In other words, *the configuration is constructible in* 30 *different ways by the same operation applied to different* 4-*sets as base.*

C.—The 30 sets fall into six such classes of 5 sets each that each class involves but 5 different ranges, which are so related that each range of a given class is found by O_{pr} to be the associate range of the remaining 4 ranges of the class. *Into the composition of ${}_{15}C_{pr}$ there accordingly enter six quintic configurations of ranges,*

$$ {}_{15}C_{pr}\begin{cases} aa,\ ab,\ ac,\ ad,\ ae;\ {}_1Q_{pr}, \\ aa,\ ab,\ ac,\ ad,\ ae;\ {}_2Q_{pr}, \\ ba,\ ba,\ bc,\ bd,\ be;\ {}_3Q_{pr}, \\ ca,\ ca,\ cb,\ cd,\ ce;\ {}_4Q_{pr}, \\ da,\ da,\ db,\ dc,\ de;\ {}_5Q_{pr}, \\ ea,\ ea,\ eb,\ ec,\ ed;\ {}_6Q_{pr}, \end{cases} $$

such that, any 4 *ranges of a Q being given, the operation of finding their associate range effects simultaneously the construction of all the Q's.*

Linear transformation properties of ${}_{15}C_{pr}$ and of analogous configurations to be presently introduced will be considered at a later stage.

Quintic configurations of point congruences. These configurations being reciprocal to the point range configurations just considered, their determination and that of their properties result immediately on replacing the notions of point, range, congruence, and lineoid, respectively, by those of lineoid, congruence, range, and point. This exchange of notions being effected, we have, in ${}_6T_{pc}$,

THEOREM (b).—*In point (lineoid)* 4-*space, any four given independent point congruences uniquely determine a fifth associate point congruence. The five congruences compose a quintic configuration of congruences such that any four of the latter determine the fifth as their associate.*

The mentioned exchange of element notions converts O_{pr} into the operation O_{pc} for finding the associate congruence of four given ones, ${}_{15}C_{pr}$ is converted into

a configuration ${}_{15}C_{pc}$ of 15 congruences which determine 15 lineoids such that *each lineoid contains* 3 *of the congruences and each congruence lies in* 3 *of the lineoids.* The congruences common to a lineoid determine a point in it. O_{pc} presents ${}_{15}C_{pc}$ in 30 different ways. ${}_{15}C_{pc}$ contains six quintic configurations ${}_iQ_{pc}$ ($i = 1, \ldots, 6$) of point congruences. The operation that determines the associate congruence of any four elements of one quintic constructs at the same time all the quintics.

Quintic configurations of sphere ranges. By means of the one-to-one correlation initially assumed between 4-space and 3-space elements, the sphere correlates of the foregoing relationships admit of immediate statement. We have, namely, in ${}_6T_{sr}$,

THEOREM (b).—*In 3-space, any four given independent sphere ranges uniquely determine a fifth associate sphere range. The five ranges constitute a quintic configuration of ranges of which any one is determined by the other four as their associate.*

The definition of the operation O_{sr} for finding the associate sphere range of a set of 4 such ranges results on replacing in the definition of O_{pr} the notion of point by that of sphere. *Operating on a 4-set of independent sphere ranges, O_{sr} presents a configuration ${}_{15}C_{sr}$ of* 15 *sphere ranges which, by intersection, determine* 15 *spheres such that each of the spheres lies in* 3 *of the ranges and each of the ranges contains* 3 *of the spheres.* The ranges of a triplet of ranges containing a same sphere lie in and determine a sphere complex, there being so determined 15 such complexes, reciprocal to the 15 spheres. ${}_{15}C_{sr}$ contains six symmetrically disposed quintic configurations ${}_iQ_{sr}$ ($i = 1, \ldots, 6$) of ranges such that under O_{sr} any range of a Q is the associate of the remaining ranges of that Q, and that the operation of constructing the associate of any one of the 30 4-sets of independent ranges constructs at once all the Q's, and therewith ${}_{15}C_{sr}$, simultaneously, in 30 ways.

Quintic configurations of sphere congruences. The principle of duality at once yields, in ${}_6T_{sc}$,

THEOREM (b).—*In 3-space, any four given independent sphere congruences uniquely determine a fifth associate sphere congruence, the five congruences constituting a quintic configuration of congruences such that any four of them determine the remaining one as their associate.*

The operation O_{sc} for finding the associate of a 4-set is *in abstractu* identical with O_{sr} and differs from it only in that it operates on reciprocal sets of elements. In finding the associate of a 4-set of congruences, O_{sc} presents a configuration ${}_{15}C_{sc}$ of 15 congruences. Of these any pair that have not a range in common lie in and so determine a sphere complex. *In this way* 15 *complexes are determined such that each complex contains* 3 *of the congruences and each of the congruences is contained in* 3 *of the complexes.* The congruences of any one of the triplets of congruences lying in a complex have one and but one sphere in common. ${}_{15}C_{sc}$ affords 30 4-sets of independent congruences, distributed so as to form six quintic configurations ${}_iQ_{sc}$ $(i = 1, \ldots, 6)$ of congruences. The Q's enter symmetrically into ${}_{15}C_{sc}$. Any Q is constructible by O_{sc} on each of 5 distinct 4-sets as basis, and the construction of any one of the Q's involves the construction of all of them.

III. *Reciprocal and Other Aspects of the Foregoing Configurations and Relationships.*

Four-space may as well be conceived primarily as a plenum of lineoids instead of points. In that case the elements that one naturally considers are: the lineoid complex, composed of the ∞^3 lineoids that generate or envelope a point; the lineoid congruence, or hyperpencil, composed of the ∞^2 lineoids common to two complexes; the lineoid pencil, composed of the ∞ lineoids common to three independent complexes. Analogously, if 3-space be thought as primarily a plenum of sphere complexes instead of spheres, one would naturally deal with the elements: complex of sphere complexes, or the ∞^3 sphere complexes that contain, or envelope, a same sphere; the congruence of sphere complexes, or the ∞^2 sphere complexes common to two complexes of complexes; the pencil of sphere complexes, composed of the ∞ sphere complexes common to three independent complexes of sphere complexes.

Such elements being employed, one would be at once led to a body of propositions constituting in generality and detail a *reciprocal* conception of the doctrine presented in section II. One or two examples of such reciprocal statements must suffice. The reciprocal aspect of theorem (a), in ${}_6T_{pr}$, is: *Given three independent lineoid congruences, there is always one and but one other lineoid congruence which taken with any one of the given congruences constitutes a pair lying in and determining a lineoid complex.* From this follows that the reciprocal aspect of theorem (b), in ${}_6T_{pr}$, is: *In lineoid 4-space, any four given independent lineoid con-*

gruences uniquely determine a fifth associate congruence, the quintic configuration of congruences being such that any four determine the fifth as their associate.

These two statements will afford a sufficient clew to the parallelization in question, which readily admits of extension to sphere theory in ordinary space.

Hypersheaves of point and sphere congruences and ranges. The assemblage of ∞^4 sphere congruences having a common sphere, and the like assemblage of point congruences having a common point, may conveniently be named, respectively, hypersheaf of sphere congruences and hypersheaf of point congruences. Analogously, their respective reciprocals, namely, the ∞^4 sphere ranges in a sphere complex and the corresponding assemblage of point ranges in a lineoid, may be, respectively, termed hypersheaf of sphere ranges and hypersheaf of point ranges. Two hypersheaves of point or sphere congruences determine as their intersection a hyperpencil (congruence) of point or sphere congruences. Reciprocally, two hypersheaves of point or sphere ranges intersect in a hyperpencil (congruence) of point or sphere ranges. From the point of view here assumed, one readily discovers a tissue of interesting relationships of which the following will serve as illustrations: *In point* $\frac{\text{4-space}}{\text{3-space}}$ *any four given independent hyperpencils* (*congruences*) *of* $\frac{\text{point}}{\text{sphere}}$ *congruences uniquely determine a fifth associate hyperpencil* (*congruence*) *of* $\frac{\text{point}}{\text{sphere}}$ *congruences. The quintic configuration of such hyperpencils is such that any four of them determine the remaining one as their associate.*

In point $\frac{\text{4-space}}{\text{3-space}}$ *any four given independent hyperpencils* (*congruences*) *of* $\frac{\text{point}}{\text{sphere}}$ *ranges uniquely determine a fifth associate hyperpencil* (*congruence*) *of* $\frac{\text{point}}{\text{sphere}}$ *ranges. The five hyperpencils are such that any four of them determine the remaining one as their associate.*

The operation of finding the associate of* 4 *given hyperpencils presents: in case of the former proposition, a configuration of* 15 *hyperpencils of* $\frac{\text{point}}{\text{sphere}}$ *congruences which determine* 15 *hypersheaves of* $\frac{\text{point}}{\text{sphere}}$ *congruences such that each of the hyper-*$\frac{\text{sheaves contains}}{\text{pencils is contained in}}$ *three of the hyper* $\frac{\text{pencils}}{\text{sheaves}}$*; in case of the latter proposition, a configuration of* 15 *hyperpencils of* $\frac{\text{point}}{\text{sphere}}$ *ranges which determine* 15 *hypersheaves*

* Of which the definition is readily found from that of O_{pr} or its correlates.

of $\genfrac{}{}{0pt}{}{\textit{point}}{\textit{sphere}}$ *ranges such that each of the hyper* $\genfrac{}{}{0pt}{}{\textit{sheaves contains}}{\textit{pencils is contained in}}$ *three of the hyper* $\genfrac{}{}{0pt}{}{\textit{pencils}}{\textit{sheaves}}$ *of ranges.*

The line and plane, the circle and (?), *of intuition.* The intuitive elements, line, plane, and circle, serving, respectively, as carriers or bases of the point range, point congruence, and sphere range, obviously may be substituted throughout the foregoing discussions for these latter elements. This is not to say, however, that the two sets of elements are identical, for such is not the case,* a fact rather strikingly evident—to name a single ground—in the failure of intuition to present any element † whatever related to the sphere congruence as, e. g. the plane of intuition is related to the congruence of points. It is accordingly of some logical interest, if not of geometric importance, that the mentioned substitution does not preserve the theories in question: it yields strictly new theories that are definitely correlated with and tend, notably in case of the line and plane as elements, through habitual association, to fuse with the old ones.

Connection with theorems by Darboux and Stephanos. By replacing in theorem (a), of ${}_6T_{sr}$, the notion of sphere range by that of the *carrier* circle, we obtain the following proposition enunciated by Darboux: ‡ *There is always one and but one fourth circle that intersects each of three given independent circles of ordinary space in two points.* The same theorem is derivable from theorem (b) by replacing the notion of sphere congruence by that of its associate *orthogonal* circle. It is noteworthy that in the Darboux proposition the elements of intersection may be *imaginary*, while in case of theorems (a) and (b), the intersections are always *real.*

The substitution in theorem (b), of ${}_6T_{sr}$ of the notion, carrier circle, for that of sphere range or in theorem (b), of ${}_6T_{sc}$, of the notion, orthogonal circle, for that of (its defining) sphere congruence, yields the beautiful propositions of Stephanos § respecting the (by him so called) *pentacycle: To every system of four circles of space there is attached a fifth of which the coordinates depend linearly upon those of the given circles. The construction of the fifth circle leads to a configuration of* 15 *circles which determine* 15 *spheres such that each sphere contains three of the circles and each circle lies on three of the spheres.*

* Cf. Poincaré: *Le continu mathematique. Revue de Metaphysique et de Morale*, Vol. I.

† The circle that is orthogonal to all the spheres of a given congruence and is thus uniquely determined by, and associable with, the congruence, plainly does not satisfy the requirement.

‡ *Sur une nouvelle définition de la surface des ondes. Comptes rendus*, Vol. XCII.

§ *Sur une configuration remarquable de cercles dans l'espace.* Comptes rendus, Vol. XCIII.

IV. *Some Covariant and Group Properties of the Configurations C and Q.*

In this closing section the term "element" e will serve to denote indifferently point range or congruence, sphere range or congruence, line, plane, circle, or other analogous element. The five equations,

$$(t) \; \ldots\ldots\ldots \; x'_j = \sum_1^5 a_{ji}\, x_i \qquad (j = 1, \ldots, 5).$$

will serve to define the general linear (homographic) point or lineoid transformation of 4-space or the general linear sphere or sphere complex transformation of 3-space according as the variables x_i are regarded as homogeneous point or lineoid or sphere or sphere complex coordinates. As (t) contains 24 disposable constants while an "element" depends on six parameters, by means of (t) four given independent "elements" are convertible into any specified second set of like independents. Let e_k and e'_k $(k = 1, \ldots, 4)$ be any two such sets, and denote by e_5 and e'_5, respectively, the associate "elements" of the given sets. From the character of the operation (cf. definition, e.g. of O_{pr}) for constructing the associate of a given set, it is plain that any transformation converting e_k into e'_k converts e_5 into e'_5. Hence,

THEOREM (c).—*The associate "element" of four given independent elements is a covariant of the given set under homographic* transformation of the containing space.*

It is readily seen, too, that a transformation converting a 4-set e_k into a 4-set e'_k likewise converts the configuration ${}_{15}C_e$, connected (as above seen) with the former set, into the configuration ${}_{15}C_{e'}$, connected with the latter. Accordingly, *any one of the* 30 *4-sets involved in a C being given, the remaining* 11 *"elements" of the C are covariants of the given set.*

Any given configuration ${}_{15}C_e$ (for concrete example, cf. ${}_{15}C_{pr}$ of III) is transformable into itself by a group G_{720} linear transformations. Given any one of the quintics ${}_iQ$ entering the C, there are six subgroups G_{120} which, respectively, convert that Q into itself and into each of the remaining Q's. The effect of other subgroups is easily made out by inspection of the foregoing table for ${}_{15}C_{pr}$.

Reciprocal transformation, exchanging points and lineoids or spheres and sphere complexes, will, of course, interchange reciprocal C's in pairs, corresponding covariant and group properties remaining essentially unaltered.

* In case of circles of ordinary space, Stephanos has pointed out that any circle of a given pentacycle is a covariant of the remaining four also under transformation by reciprocal radii vectores.

Some Relations Between Number Theory and Group Theory.

BY G. A. MILLER.

The $\phi\left(\frac{m}{k}\right)$ numbers* which are less than m and have the same highest common factor (k) with m become an abelian group (G) when they are combined by multiplication modulo m, and G is the group of isomorphisms of the cyclic group of order $\frac{m}{k}$.† Every possible finite multiplication group in which the operators are represented by numbers is either such a group or it is contained in such a group.‡ The group G may be made simply isomorphic with the group of isomorphisms of the cyclic group of order $\frac{m}{k}$ by associating with each operator of the latter, k into the index of the power into which this operator transforms every operator of the cyclic group of order $\frac{m}{k}$.

The identity of G is, therefore, its number which is congruent to unity modulo $\frac{m}{k}$. In fact, the numbers of G constitute the same multiplication group with respect to modulus $\frac{m}{k}$ as they constitute with respect to modulus m. The number of invariants‖ in G is therefore equal to the number of different odd primes which divide $\frac{m}{k}$ increased by β_0, where $\beta_0 = 2$, 1, or 0 as $\frac{m}{k}$ is of the form $8n$, $8n+4$, or $\not\equiv 0 \bmod 4$ respectively. Each of these invariants is even since $\phi(l)$ is even whenever $l > 2$. That $\phi(l)$ is even

* Only positive integers are considered in this article.

† Annals of Mathematics, Vol. 2 (1901), p. 77.

‡ Ibid., Vol. 6 (1905), p. 44.

‖ That is, the smallest possible number of independent generators of G.

follows from the fact that it represents the number of operators of highest order (the number of generators) in the cyclic group of order l. Since we may associate an operator with its inverse and this property is reciprocal, it follows that the number of operators of order r, $r > 2$, in any group is even. In particular, the number of generators of any cyclic group whose order exceeds 2 is even.

It may be observed that this proof of the elementary theorem that $\phi(l)$, $l > 2$, is even associates it with the fundamental theorem that the number of operators of order l in any group is even, and hence the number of operators of order 2 in any group of even order is odd. The main object of the present paper is to exhibit relations between fundamental theorems of the two subjects mentioned in the heading. Only one more will be noted here. Let $d_1, d_2, \ldots, d_\lambda$ be the orders of all the cyclic subgroups (including the identity) of any group of order g. Since the total number of operators of highest order contained in all of these subgroups is equal to the order of the group, it follows that

$$g = \phi(d_1) + \phi(d_2) + \ldots + \phi(d_\lambda)$$

when the group is cyclic the numbers $d_1, d_2, \ldots, d_\lambda$ are all distinct and represent all the divisors of g. In this special case the formula reduces to the well known theorem that the totient of a number is the sum of the totients of its divisors.

§1. *Quadratic Residues of Numbers.*

The necessary and sufficient condition that G is cyclic is that $\frac{m}{k}$ has primitive roots.* That is, G is cyclic only when $\frac{m}{k}$ has one of the following values: $4, p^a, 2p^a$, p being an odd prime. In each of these cases G involves only one operator of order 2 and hence just half of its operators have square roots under G. Each of these operators has just two square roots. These facts follow directly from the elementary theorem that the n^{th} powers of all the operators of any abelian group constitute a quotient group, which is the entire group whenever n is prime to the order of the group, and that *each operator of this quotient group may be made to correspond to all its n^{th} roots in an isomorphism between the*

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* The number m' has primitive roots whenever the group of isomorphisms of the cyclic group of order m' is cyclic and vice versa.

group and the given quotient group. In particular, each operator that has n^{th} roots has the same number of such roots.

If β represents the smallest number of independent generators of G, the operators of order 2 in G generate a group of order 2^β. Hence it results from the given theorem that just $\frac{1}{2^\beta}$ of the numbers of G have square roots and that each number that has one square root has just 2^β square roots, modulo m or modulo $\frac{m}{k}$. The manner in which the value of β may be obtained from the number $\frac{m}{k}$ was noted above. In the language of number theory this result is generally stated as follows: The congruence

$$x^2 \equiv N \text{ mod. } \frac{m}{k}, \text{ or mod. } m$$

where N and $\frac{m}{k}$ are relatively prime, has either no solution or it has 2^β solutions, β being equal to the number of odd prime factors of $\frac{m}{k}$ increased by β_0.

The group of isomorphisms (I) of the cyclic group of order 2^n is the direct product of a cyclic group of order 2^{n-2}, which is composed of all the operators of I which are commutative with operators of order 4 in the cyclic group of order 2^n, and the group of order 2 generated by the operator of I which transforms each operator of this cyclic group into its inverse.* The square of every operator of I is therefore also the square of an operator in the given cyclic subgroup of order 2^{n-2}. Since all the operators of the latter are commutative with the operators of order 4 in the cyclic group of order 2^n their squares must be commutative with the operators of order 8 in this cyclic group. Hence these squares must transform each operator into a power which is congruent to unity modulo 8. Since the operators of I transform the operators of the cyclic group of order 2^n into every odd power, it follows that *the square of every odd number is congruent to unity modulo* 8.

Since the squares of the operators of the given cyclic subgroup of order 2^{n-2} give all the operators of I which are commutative with the operators of order 8 in the cyclic group of order 2^n, it follows that every odd number which

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* Bulletin of the American Mathematical Society, Vol. 7 (1901), p. 351.

is congruent to unity modulo 8 is the square of some other odd number modulo 2^n. That is, every such odd number is a quadratic residue modulo 2^n. Since the identity of I is its own square it follows that every odd number is a quadratic residue of 2 and every odd number congruent to unity modulo 4 is a quadratic residue of 4.

In general, the 2^δ, $\delta > 0$, power of every operator of I is the 2^δ power of an operator in the given cyclic subgroup of order 2^{n-2} and these powers constitute all the operators of I which are commutative with each operator of the subgroup of order $2^{\delta+2}$ in the given cyclic group of order 2^n. That is, *every number which is congruent to unity modulo $2^{\delta+2}$ is the 2^δ power of some number modulo 2^n, where n is arbitrary.* Conversely, the 2^δ power of every odd number is congruent to unity modulo $2^{\delta+2}$. If a number has one 2^δ root it has just $2^{\delta+1}$ such roots modulo p^α, $\alpha > \delta + 1$, since I is the direct product of a cyclic group of even order and an operator of order 2.

The preceding results apply directly to the odd multiples of any odd number (k). The first 2^{n-1} of these multiples constitute a group which is simply isomorphic with I if the products are taken modulo 2^nk. Hence each of these numbers which is congruent to unity modulo $2^{\delta+2}$ is the 2^δ power of some number modulo 2^nk where n is arbitrary. That is, any number which is congruent to unity modulo $2^{\delta+2}$ is the 2^δ power of some number modulo 2^nk, where k is any factor of the first number and n is arbitrary.

The group of isomorphisms I of the cyclic group (C) of order p^α (p being an odd prime) is the direct product of two cyclic groups of order $p^{\alpha-1}$ and $p - 1$ respectively. The former is composed of all the operators of I which transform the operators of C into powers whose indices are congruent to unity modulo p.* In other words, when I is represented as a number group modulo p^α the numbers which correspond to this subgroup of order $p^{\alpha-1}$ are composed of all the numbers less than p^α which are congruent to unity modulo p. The p^δ powers of these numbers are congruent to unity modulo $p^{\delta+1}$ and are composed of all the numbers modulo p^α which have this property. Any number which is congruent to unity modulo $p^{\delta+1}$ is, therefore, the p^δ power of some other number modulo p^α, where α is arbitrary.

If an operator of I corresponds to a number which is incongruent to unity

* Bulletin of the American Mathematical Society, Vol. 7 (1901), p. 351.

modulo p it must transform every operator of C besides the identity. In particular, all the operators of the given cyclic subgroup of order $p-1$ transform every operator of C besides the identity. If any power of such an operator is commutative with an operator of order p in C it must be commutative with every operator of C; i.e. it must be the identity. In other words, the numbers which correspond to the operators of this cyclic group of order $p-1$ belong to the same exponent with respect to each of the moduli p, and p^α. If such a number is an r^{th} root modulo p it must also be an r^{th} root modulo p^α, and it must have just r such roots with respect to each modulus since I is cyclic. From this it follows directly that when r is prime to p *every number which is an r^{th} root modulo p is also an r^{th} root modulo p^α, α being arbitrary.* In particular, a quadratic residue of p is also a quadratic residue of p^α, and it is the square of just two numbers with respect to each modulus.

This result may also be seen as follows: If a number is an r^{th} root modulo p it must correspond to an operator of I whose $\frac{p-1}{r}$ power is a power of p. If this condition is satisfied it is clearly also an r^{th} root of p^α. In other words, the operators of I which are r^{th} powers have for their constituent whose order is prime to p an operator whose order divides $\frac{p-1}{r}$, and vice versa. As this condition is independent of the value of α it furnishes a direct proof of the theorem in question.

We shall next consider the quadratic character of -1 with respect to modulus m. This number corresponds to the operator of I which transforms every operator of the cyclic group of order m into its inverse. This operator is clearly of order 2 and hence can only be the square of an operator of order 4. Moreover, when I is cyclic and involves an operator of order 4, its square must be the operator of order 2 contained in I. Hence, when m is either p^α or $2\,p^\alpha$ the necessary and sufficient condition that -1 is a quadratic residue is that $p^{\alpha-1}(p-1)$ is divisible by 4; i.e. $p-1$ must be divisible by 4.

It has been observed above that the operator of I which transforms each operator of the cyclic group of order 2^n into its inverse may be used as an independent generator of I and hence cannot be a power of any operator of higher order contained in I. In particular, -1 is a non-quadratic residue of 2^n, $n>1$, when $n=1$, $-1\equiv 1$ and hence may be regarded as a quadratic residue of every odd number. In general, I is the direct product of the Sylow subgroups

of the cyclic group of order m. Hence -1 is a quadratic residue of m only when it is a quadratic residue of the orders of these Sylow subgroups. That is, *the necessary and sufficient condition that -1 is a quadratic residue of m is that all the odd prime factors of m are of the form $4l+1$ and that m is not divisible by 4.* When this condition is satisfied the operator of order 2 in I which transforms each operator of the cyclic group of order m into its inverse is the square of an operator of order 4 and vice versa.

Since -1 corresponds to an operator of order 2 in I it is very easy to determine its general root character modulo m. When I is cyclic this operator has r^{th} root whenever the order of I is divisible by $2r$; i.e. whenever $p^{a-1}(p-1)$ is divisible by $2r$. In general, the necessary and sufficient condition that -1 has r^{th} roots modulo $m = 2^{a_0}, p_1^{a_1}, p_2^{a_2} \ldots\ldots p_\lambda^{a_\lambda}$ is that $2r$ divides each of the numbers $p_a^{a_a-1}(p_a-1)(\alpha = 1, 2, \ldots\ldots, \lambda)$ and that $a_0 = 1$ or 0 whenever r is even. Since -1 is any odd root of itself this condition has little meaning unless r is even.

§2. *Proof of several other fundamental theorems from the standpoint of group theory.*

From the introductory remarks it follows that the numbers (elements) of a number group G are either all odd or all even whenever the modulus m is even. When m is odd the $\phi\left(\frac{m}{k}\right)$ elements of G include just as many even numbers as odd numbers. In the subgroups of G the number of the even elements need not be the same as that of the odd elements unless the subgroup involves -1 modulo $\frac{m}{k}$. These theorems follow directly from the fact that the product of -1 into an even element is odd and vice versa. As illustration of subgroups of G in which the number of even elements is not equal to the number of odd elements, we give the following four groups:

1, 2, 4, 8 mod. 15; 1, 4, 7, 13 mod. 15; 7, 28, 49, 91 mod. 105; 1, 2, 4, 8, 11, 16 mod. 21.

From the fact that -1 is of order 2 it follows that each number which has a square root with respect to an odd modulus must have an equal number of odd and even numbers as square roots. If the modulus is even all the square roots of odd numbers are odd and those of even numbers are even since the elements of such a group are either all even or all odd.

Some of the formulas relating to the totient of m can be readily obtained by means of group concepts. Whenever m is not a power of a prime it can be resolved into two factors m_1, m_2 which are relatively prime. In this case the cyclic group of order m is the direct product of the two cyclic groups of orders m_1, m_2 respectively. The operators of highest order in the cyclic group of order m are obtained by multiplying the operators of highest order in the cyclic group of order m_1, into those of highest order in the cyclic group of order m_2. This furnishes a direct proof of the formula

$$\phi(m) = \phi(m_1)\,\phi(m_2)$$

whenever m_1, m_2 are relatively prime and $m = m_1 m_2$. Moreover, all the subgroups of the cyclic group of order p^a are contained in its subgroup of order p^{a-1}. That is, $\phi(p^a) = p^a - p^{a-1} = p^a\left(1 - \frac{1}{p}\right)$. Euler's ϕ-function of m is a direct consequence of these two results.

The total number of different sets of k numbers such that none of these numbers exceeds m and that the greatest common divisor of m and all the numbers of any set is unity has been denoted by $\phi_k(m)$ and is called the totient of order k with respect to m.* When $k = 1$ it reduces to the ordinary totient and the subscript is generally omitted. The value of $\phi_k(m)$ may be determined as follows:

Suppose that an abelian group G has k independent generators $(s_1, s_2, \ldots, s_k)$ each being of order m. It is well known that every operator of G can be written in the form

$$s = s_1^{a_1},\ s_2^{a_2} \ldots\ s_k^{a_k};\quad a_1, a_2, \ldots, a_k = 1, 2, \ldots, m.$$

The order of s is m whenever the greatest common divisor of the numbers $m, a_1, a_2, \ldots, a_k$ is unity and vice versa. Hence, $\phi(m)$ is the number of operators of order m in G. As methods are known to find the total number of operators of any order in an abelian group† the determination of $\phi_k(m)$ becomes a very special case of these general methods.

If m is not a power of a prime an independent generator of order m may always be replaced by two independent generators of orders m_1, m_2 respectively,

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* Jordan, Traite des substitutions, 1870, p. 96; cf. Cahan, Théorie des nombres, 1900, p. 36.

† Zsigmondy, Monatshefte für Mathematik und Physik, Vol. 7 (1896), p. 227; cf. Annals of Mathematics, Vol. 6 (1904), p. 3.

where $m_1\, m_2 = m$ and m_1, m_2 are relatively prime. Resolving each of the independent generators of G into two such factors, it is evident that G is the direct product of two subgroups having m independent generators of orders m_1, m_2 respectively. Moreover, the number of operators of highest order in G is equal to the product of the numbers of operators of highest order in each of these subgroups. In other words,

$$\phi_k(m) = \phi_k(m_1)\,\phi_k(m_2)$$

whenever $m = m_1\, m_2$ and m_1, m_2 are relatively prime.

When $m = p^a$ the number of operators of order p^a in G is equal to the total number of its operators (p^{ak}) diminished by $p^{(a-1)k}$, the number of its operators whose orders divide p^{a-1}. That is, $\phi_k(p^a) = p^{ak} - p^{(a-1)k} = p^{ak}\left(1 - \frac{1}{p^k}\right)$. From this and the preceding formula the value of the function $\phi_k(m)$ can be directly obtained. Hence,

$$\phi_k(m) = m^k\left(1 - \frac{1}{p_1^k}\right)\left(1 - \frac{1}{p_2^k}\right) \cdots \left(1 - \frac{1}{p_\lambda^k}\right)$$

$p_1, p_2 \cdots p_\lambda$ being the distinct prime factors of m.

One of the earliest developments in number theory relates to perfect numbers. The order (m) of a cyclic group (M) is said to be perfect whenever M is such that the sum of the orders of all its subgroups is equal to m. It is easy to see that such a cyclic group cannot contain a subgroup whose order (d) is perfect. Since M would contain a subgroup of order $\frac{m}{d}$ into every divisor of d (excluding d itself) and since the sum of these orders would be md, M could contain no other subgroup if m and d were both perfect. As this set of subgroups does not include the identity, it follows that the order of a cyclic group is redundant whenever the order of one of its subgroups is perfect.

It should be added that the developments of this article have close contact with those of Zsigmondy, "Beiträge zur Theorie Abel'scher Gruppen und ihrer Anwendung auf die Zahlentheorie," Monatshefte für Mathematik und Physik, Vol. 7 (1896), pp. 185-289, and the note on "Holomorphisms and primitive roots," Bulletin of the American Mathematical Society, Vol. 7 (1901), pp. 350-354. The object of the present article is to exhibit certain additional developments where the group concept seems especially useful in the study of number theory.

STANFORD UNIVERSITY, *March* 1905.

The Differential Invariants of Space.

By J. E. Wright.

The object of this paper is to solve the three following problems:

(I) The determination of the differential invariants of all orders, of space of any number of dimensions.

(II) The determination of invariants of any manifold in this space under any transformation which leaves its "shape" unaltered.

(III) The determination of the "deformation" invariants of any manifold in this space.

In the statement of the problems, the word "invariant" is used to include the whole class of Gaussian Invariants, Parameters and Covariants. The problems are considered solved when a method is given for determining a complete functionally independent set of invariants by direct processes.

In problem (I) for example, it will be shown that all the invariants may be expressed as the algebraic invariants of certain forms, and a method will be given for writing down these forms in succession. At one stage of the work it is found simpler to discard certain of these forms and to introduce instead a complete set of invariants due to them.

In (II) the solutions are given as the invariants of algebraic forms, with the exception of those corresponding to the set introduced in (I).

The parameters which arise in the solution of (III) are also expressed in terms of algebraic invariants, but for the determination of the Gaussian invariants, and the covariants, it is found simpler to make use of a method whereby they are expressed in a different manner. The method in question leads to the complete set of invariants in every case, but the expression of the solutions as algebraic invariants has many advantages from some points of view. The chief advantage is perhaps that the well known parameters are immediately recognized.

(III) is really equivalent to the determination of the invariants of a quadratic differential form. There is indeed an apparent limitation on the generality of the form, but it is readily seen that the generality of the results is not thereby affected, and that in fact the problem of the determination of a functionally independent complete set of invariants for a quadratic form is also solved. Lie* suggested this problem for the differential form in two variables. Using Lie's method Žorawski† investigated the invariants of the first and second orders for a general quadratic form. He also considered the question of the number of functionally independent invariants of any order. This question of number has engaged the attention of several investigators, but the complete results are given by C. N. Haskins.‡ His method does not, however, lead to the expressions for the invariants themselves. Forsyth || has obtained the invariants of the first, second, and third orders, of space of three dimensions. His method is, however, unsuitable for the determination of invariants of higher orders. He has also § obtained the invariants of the first, second, and third orders which are of the type sought in (II) for a surface in space of three dimensions.

Maschke,** by the use of symbolic methods has developed a process for determining invariants of any order from known invariants. But his work does not suggest any obvious method of determining a complete set.

§1. The method pursued in this paper for the solution of problem (I) consists essentially in a factorisation of the problem into two others, each of which can be readily solved. Take any system of variables $u_1, u_2 \ldots\ldots u_m$, and let them denote a system of curvilinear coördinates in ordinary space of m dimensions. If the square of the linear element be

$$ds^2 \equiv \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs} du_r du_s,$$

where a_{rs} is a function of the u's, it must be possible to find $x_1, x_2 \ldots\ldots x_m$, functions of the u's, such that

$$ds^2 = dx_1^2 + dx_2^2 + \ldots\ldots dx_m^2.$$

* Ueber Differentialinvarianten. Math. Ann., Vol. XXIV (1884), pp. 574-575.

† Ueber Biegungsinvarianten, Acta Mathematica, Vol. XVI (1892-93), pp. 1-64.

‡ Trans. Amer. Math. Soc., Vol. III (1902), pp. 71-91; also Trans. Amer. Math. Soc., Vol. V (1904), pp. 167-192.

|| Philosophical Transactions, Series A, Vol. 202 (1903), pp. 277-333.

§ Philosophical Transactions, series A, Vol. 201 (1903), pp. 329-402.

** Trans. Amer. Math. Soc., Vol. I (1900), pp. 197-204; and Vol. IV (1903), pp. 445-469.

It is easy to prove that if one set of functions $x_1 \ldots\ldots x_m$ is given, the most general possible set is given by performing a general orthogonal transformation and a translation on the set x.

In fact if $x_1, x_2 \ldots\ldots x_m$ and $x'_1, x'_2 \ldots\ldots x'_m$ are two different sets, there must exist the relations

$$x'_i = \beta_i + \sum_{j=1}^{m} a_{ij}\, x_j, \qquad (i = 1, 2 \ldots\ldots m)$$

where the a's and β's are constants such that

$$\sum_{k=1}^{m} a_{ik}\, a_{jk} = 0 \qquad (i \neq j)$$

$$\sum_{k=1}^{m} a_{ik}^2 = 1\,.$$

Now any invariant will be a function of the a's and their derivatives. It will, therefore, be a function of the x's and their derivatives. This function must be invariant under the most general orthogonal transformation and under the most general translation.

It is now easy to see that the problem considered separates into two parts:

(A) The determination of all invariants under a general transformation on the u's, and subject to the condition that the x's are invariant.

(B) The selection from these of those functions which are invariantive when a translation and an orthogonal transformation are performed on the x's.

The first of these is equivalent to the determination of all the invariants of any number of functions of the set of variables $u_1, u_2 \ldots\ldots u_m$. Let these functions be $f^{(1)}(u_1, u_2 \ldots\ldots u_m)$, $f^{(2)}(u_1, u_2 \ldots\ldots u_m) \ldots\ldots f^{(r)}(u_1, u_2 \ldots\ldots u_m)$. We take as the variables occurring in the invariants

(1) all possible derivatives of the f's.

(2) $u_1, u_2 \ldots\ldots u_m$.

(3) $du_1, du_2 \ldots\ldots du_m, d^2u_1, d^2u_2 \ldots\ldots d^2u_m$, etc.

It is true, as pointed out by Forsyth,* that we may take account of the ratios of the set of variables (3), by introducing equations of the type $\phi(u_1u_2 \ldots\ldots u_m) = 0$. It is, however, simpler to preserve these variables.

* Philosophical Transactions, series A, Vol. 201 (1903), p. 333.

There is one simplification which may be made; there is no need to take account of more than m functions f, for any other function may be expressed in terms of the first m of them. Hence any derivative of this function may be expressed in terms of the functions $f^{(1)}, f^{(2)} \dots f^{(m)}$ and their derivatives. It, therefore, follows that any invariant involving this function can be expressed in terms of $f^{(1)}, f^{(2)}, \dots f^{(m)}$ and their derivatives. The invariant is, therefore, reducible.

To solve the problem (A), we make use of Lie's* method, with slight modifications in the details of the work. The method is the following: Let F be any invariant involving the variables specified. On this we perform the most general transformation of the group of point transformations of the variables u, and assume that it becomes F'. Then the condition for invariance is $F' = \Omega^{\mu} F$, where Ω is the Jacobian of the transformation, and μ is a number.

Let $\frac{dF}{dt}$ denote the effect on F of the most general infinitesimal operator of the group, then Ω^{μ} becomes $1 + \mu\left(\sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r}\right)\delta t$, where $\frac{du_r}{dt} = \xi_r \; (r = 1, 2, \dots m)$,

and
$$\frac{dF}{dt} = \mu\left\{\sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r}\right\} F. \tag{1}$$

If F satisfies this equation for all possible operators, then it is an invariant of the type desired. Now since the group is the most general in the variables u, the ξ's are arbitrary functions of their arguments, and hence F must be a function satisfying the system of equations obtained by equating to zero the coefficients in (1) of the various derivatives of the ξ's. As proved by Lie (l.c.) the system of equations thus obtained is complete, and therefore the number of functionally independent solutions is $M - N$, where M is the number of variables, and N the number of linearly independent equations.

To determine the equation system we require the increments under the infinitesimal transformation of the variables in F. The detailed expressions for these increments are not needed for the present work, but Forsyth's† method for their determination is preferable to Lie's, and is therefore used.

Let u_i denote the original, and u'_i the transformed variables, and let $u_i + k_i$ become $u'_i + k'_i$, $(i = 1, 2 \dots m)$.

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*Loc. cit., pp. 564-566.

†Philosophical Transactions, series A, Vol. 201 (1903) pp. 336-340.

Then $k'_i = u'_i + k'_i - u'_i$,

$$= u_i + k_i + \xi_i (u + k)\, \delta t - [u_i + \xi_i (u)\, \delta t]$$

where $\xi_i (u)$ is written for $\xi_i (u_1 u_2 \ldots . u_m)$.

Hence $k'_i = k_i + [\xi_i (u + k) - \xi_i (u)]\, \delta t$, and therefore if ϕ denotes any function of the variables u, and ϕ' its transformed,

$$\phi (u + k) = \phi' (u' + k') = \phi' (u' + k + \overline{\xi (u + k) - \xi (u)}\, \delta t)$$

$$= \phi' (u' + k) + \sum_{i=1}^{m} [\xi_i (u + k) - \xi_i (u)] \frac{\partial \phi' (u' + k)}{\partial (u'_i + k_i)} \delta t + \ldots .$$ or, if small quantities of order higher than the first are neglected

$$-\frac{d\phi (u + k)}{dt} = \sum_{i=1}^{m} [\xi_i (u + k) - \xi_i (u)] \frac{\partial \phi (u + k)}{\partial (u_i + k_i)},$$ where $\frac{d}{dt}$ operates only on u, and not on k. This equation holds for all values of the variables k, and therefore the coefficients of corresponding powers of k on both sides may be equated. In this way are obtained the increments of all the derivatives of the functions f. For our present purpose we only require those terms in the expression for any increment which involve the highest derivatives of the ξ's.

Let $f_{a_1 a_2 \ldots a_m}$ denote $\frac{\partial^n f}{\partial u_1^{a_1} \partial u_2^{a_2} \ldots . \partial u_m^{a_m}}$, where $a_1 + a_2 + \ldots . + a_m = n$.

Then

$$-\frac{d}{dt} f_{a_1 a_2 \ldots . a_m} = \sum_{i=1}^{m} (\xi_i)_{a_1 a_2 \ldots . a_m} \frac{\partial f}{\partial u_i}$$ + terms involving lower derivatives of the ξ's. We write $u'\ u'' \ldots . u^{(\lambda)} \ldots .$ for the variables du, $d^2u, \ldots . d^{(\lambda)}u, \ldots .$. The increments of these variables may be written down immediately. In fact

$$\frac{d}{dt} u_j^{(\lambda)} = \left\{ \sum_{i=1}^{m} u'_i \frac{\partial}{\partial u_i} \right\}^{\lambda} \xi_j$$ + terms involving lower derivatives of the ξ's. The expression $\left\{ \sum u'_i \frac{\partial}{\partial u_i} \right\}^{\lambda}$ is supposed expanded by the multinomial theorem, and then applied as an operator to ξ_j.

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It is convenient to classify invariants according to 'order'. An invariant is of the n^{th} order when the increments of its variables involve n^{th} derivatives of the ξ's but no higher derivatives. We call two invariants of the n^{th} order 'independent' when one cannot be expressed in terms of the other together with invariants of order less than n. Suppose that the invariants of orders up to $n-1$ are known. Then the invariants of orders up to n consist of these and a certain number of independent invariants of the n^{th} order. Let F be an invariant of the n^{th} order. It must satisfy the system of equations obtained by equating to zero the coefficients of the n^{th} derivatives of the ξ's in equation (1),

$$\sum_{k=1}^{m} \frac{\partial f^{(k)}}{\partial u_i} \frac{\partial F}{\partial f^{(k)}_{\alpha_1 \alpha_2 \ldots \alpha_m}} + \frac{n!}{\alpha_1!\,\alpha_2!\ldots\alpha_m!} (u_1')^{\alpha_1} (u_2')^{\alpha_2} \ldots (u_m')^{\alpha_m} \frac{\partial F}{\partial u_i^{(n)}} = 0. \qquad (2)$$

where $i = 1, 2 \ldots m$, and $\alpha_1\ \alpha_2 \ldots \alpha_m$ take all positive integral and zero values subject to the condition $\alpha_1 + \alpha_2 + \ldots \alpha_m = n$.

In addition, F must satisfy the equations obtained from the lower derivatives of the ξ's. It is easy to see that m independent solutions of the equations (2) are $d^n f^{(1)}, d^n f^{(2)}, \ldots d^n f^{(m)}$, where $d^n f$ denotes the n^{th} total differential of f. Also the number of derivatives of F in these equations is obviously greater by m than the number of equations. Hence we have all the functionally independent solutions in variables of the n^{th} order, provided the equations are linearly independent. One of the determinants of the matrix of the equations may be shown to be

$$\left\{ J \begin{pmatrix} f^{(1)} & f^{(2)} & \ldots & f^{(m)} \\ u_1 & u_2 & \ldots & u_m \end{pmatrix} \right\}^{\frac{(m+r-1)!}{r!\,(m-1)!}}.$$

The functions f are assumed independent, and therefore their Jacobian does not vanish. Hence the determinant mentioned does not vanish, and therefore the equations are linearly independent. Hence all the solutions of the equations (2) are obtained. It may readily be verified that each of the solutions $d^n f$ satisfies the remaining equations, and is an absolute invariant; this is also obvious from the form of the solutions. We may now use these solutions to get rid of n^{th} order variables from the remaining equations. When this is done the equations become precisely those for invariants of orders less than or equal to $(n-1)$. Continuing this process we finally arrive at the complete system of invariants,

$$d^\lambda f^{(\mu)}, \qquad \begin{pmatrix} \lambda = 2, 3 \ldots n \\ \mu = 1, 2 \ldots m \end{pmatrix},$$

together with the solutions of the system of equations for invariants of the first order. A slight modification is here necessary, owing to the fact that F occurs explicitly. It is easy to see that we have the m solutions $df^{(1)}\; df^{(2)} \; \ldots . \; df^{(m)}$, which are absolute invariants, and there yet remains one other integral, which is manifestly the Jacobian of the f's. This is a relative invariant. In fact, if the transformed variables are $U_1, U_2 \ldots . U_m$,

$$J\begin{pmatrix} f^{(1)} & \ldots . & f^{(m)} \\ U_1 & \ldots . & U_m \end{pmatrix} \times J\begin{pmatrix} U_1 & \ldots . & U_m \\ u_1 & \ldots . & u_m \end{pmatrix} = J\begin{pmatrix} f^{(1)}, f^{(2)} & \ldots . & f^{(m)} \\ u_1, u_2 & \ldots . & u_m \end{pmatrix}$$

and therefore the μ of equation (1) is -1.

Hence we have the theorem: *A complete functionally independent system of invariants of m functions f in the variables $u_1, u_2, \ldots . u_m$, involving variables up to the n^{th} order, is given by*

$$\begin{matrix} df^{(1)}, & df^{(2)}, & \ldots . & df^{(m)}, \\ d^2f^{(1)}, & d^2f^{(2)}, & \ldots . & d^2f^{(m)}, \\ \vdots & & & \vdots \\ d^nf^{(1)}, & d^nf^{(2)}, & \ldots . & d^nf^{m}, \end{matrix}$$

which are absolute invariants, together with

$$J\begin{pmatrix} f^{(1)}, f^{(2)}, & \ldots . & f^{(m)} \\ u_1, \; u_2, & \ldots . & u_m \end{pmatrix},$$

which is a relative invariant with $\mu = -1$.

The most general invariant is therefore a function of the f's and the above absolute invariants, multiplied by some power of J.

§2. We are now in a position to determine all the invariants of ordinary space of m dimensions.

We call the functions $f^{(1)}, f^{(2)}, \ldots . f^{(m)}$, $x_1, x_2, \ldots . x_m$. The invariants must be functions of

$$\begin{matrix} & J, & & \\ x_1, & x_2, & \ldots . & x_m, \\ dx_1, & dx_2, & \ldots . & dx_m, \\ d^2x_1, & d^2x_2, & \ldots . & d^2x_m, \\ \vdots & & & \vdots \\ d^nx_1, & d^nx_2, & \ldots . & d^nx_m, \end{matrix}$$

and of any number of functions $f^{(k)}(x_1, x_2, \ldots . x_m)$, and of their derivatives. They must be invariants under the most general translation, and under the most general orthogonal transformation.

As in the previous case, there is no need to consider more than m functions f, since $f^{(s)}$, $(s > m)$, can be expressed in terms of $f^{(1)}, f^{(2)} \ldots . f^{(m)}$. We may at once take account of the translation; it is equivalent to the condition that the x's do not occur explicitly. For the orthogonal transformation, the transformation scheme for the x's is

$$x_i' = x_i + \left\{ \sum_{k=1}^{m} a_{ik}\, x_k \right\} \delta t,$$

where $a_{ik} = - a_{ki}$ for all values of i and k.

We pursue the same method as before to determine the invariants. There is now, however, the important limitation that the second and higher derivatives of the increments of the x's vanish.

Let the increment of x_i be denoted by ξ_i, then $\frac{\partial \xi_i}{\partial x_i}$ is zero, and therefore the fundamental equation (1) becomes

$$\frac{dF}{dt} = 0.$$

The number of operators in $\frac{d}{dt}$ is $\frac{1}{2}\, m\,(m - 1)$, since each operator corresponds to an independent constant a. Therefore, provided the operators are unconnected, the number of solutions is $\frac{1}{2}\, m\,(m - 1)$ less than the number of variables. Now taking account of differentials and of differential coefficients as far as the n^{th} order, and assuming m functions f, the number of variables is

$$1 \text{ of type } J,$$

$$mn \text{ of type } d^r x_s,$$

$$m \left\{ \frac{(m+n)!}{m!\, n!} - 1 \right\} \text{ of type } \frac{\partial^r f}{\partial x_1^{\beta_1}\, \partial x_2^{\beta_2} \ldots . \partial x_m^{\beta_m}}.$$

Altogether there are therefore

$$\frac{(m+n)!}{(m-1)!\, n!} + (n - 1)\, m + 1 \text{ variables,}$$

and therefore the number of solutions is

$$\frac{(m+n)!}{(m-1)!\; n!} + (n-1)\, m + 1 - \tfrac{1}{2}\, m\,(m-1)\,,$$

provided the equations for the invariants are independent.

In order to obtain the equations, the increments of the variables are required. For the derivatives of the f's use is made of the equation

$$-\frac{d}{dt} f(x+k) = \sum_{i=1}^{m} \left[\xi_i(x+k) - \xi_i(x)\right] \frac{\partial f(x+k)}{\partial (x_i + k_i)}\,.$$

In this case

$$\xi_i(x+k) - \xi_i(x) = \sum_{j=1}^{m} k_j a_{ij}\,,$$

and therefore

$$-\frac{d}{dt}\,\frac{\partial^n f}{\partial x_1^{r_1},\, \partial x_2^{r_2} \ldots\ldots \partial x_m^{r_m}} = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\, r_j \frac{\partial^n f}{\partial x_1^{s_1},\, \partial x_2^{s_2} \ldots\ldots \partial x_m^{s_m}}$$

$$\left(\begin{matrix} s_i = r_i + 1\,;\; s_j = r_j - 1\,;\; s_\lambda = r_\lambda,\; \lambda \neq i,\; \lambda \neq j\,; \\ r_1 + r_2 + \ldots\ldots r_m = n \end{matrix}\right).$$

Now consider the algebraic form of the n^{th} order in the variables $X_1, X_2, \ldots\ldots X_m$,

$$A_n \equiv \left\{ \sum_{i=1}^{m} X_i \frac{\partial}{\partial x_i} \right\}^n f(x_1, x_2 \ldots\ldots x_m)\,,$$

and let the orthogonal transformation already used be performed on the variables X. The increments of the coefficients are precisely those given above. In addition we notice that the increments for the magnitudes $d^n x$ are exactly similar to those for the magnitudes x, and that J is an absolute invariant.

We therefore immediately obtain the general result:

The functionally independent set of invariants of orders up to and including the n^{th}, of space of m dimensions, are J, and the orthogonal algebraic invariants of the system of m-ary forms

$$\begin{matrix} A_1^{(1)}, & A_1^{(2)}, & \ldots\ldots & A_1^{(m)}, \\ A_2^{(1)}, & A_2^{(2)}, & \ldots\ldots & A_2^{(m)}, \\ \vdots & & & \vdots \\ A_n^{(1)}, & A_n^{(2)}, & \ldots\ldots & A_n^{(m)}, \end{matrix}$$

$$\sum_{i=1}^{m} d^r x_i\, X_i, \qquad\qquad (r = 1,\, 2 \ldots\ldots n).$$

The most general invariant of the space under a general point transformation on the variables u, is therefore a function of the functions f, and of the algebraic invariants given above, multiplied by some power of J.

It is worthy of note that the algebraic forms A, are the polar forms of the functions J. The linear forms

$$\sum_{i=1}^{m} d^r x_i \, X_i$$

may be excluded, provided there are m functions f, if certain additional invariants are taken account of. In fact, it is clear that the form

$$\sum_{i=1}^{m} d^r x_i \, X_i$$

leads to the functionally independent set of invariants

$$\sum_{i=1}^{m} d^r x_i \, \frac{\partial f^{(\rho)}}{\partial x_i}, (\rho = 1, 2 \, \ldots . \, m)$$

and it follows at once that there is no need to retain the linear forms specified, provided we add to the set of invariants the expressions

$$d^\lambda f^{(\rho)} . \qquad \begin{pmatrix} \lambda = 1, 2 \, \ldots . \, n \\ \rho = 1, 2 \, \ldots . \, m \end{pmatrix}$$

which are obviously functionally independent invariants.

It is convenient to modify the result obtained by including the quadratic form

$$\sum_{i=1}^{m} X_i^2 ,$$

and then the most general invariant is seen to be a function of

$$\begin{array}{lll} df^{(1)}, & df^{(2)}, \ldots . & df^{(m)}, \\ d^2 f^{(1)}, & \ldots . & d^2 f^{(m)}, \\ \vdots & & \vdots \\ d^n f^{(1)}, & \ldots . & d^n f^{(m)}, \end{array}$$

and of the general algebraic invariants of the forms A, and

$$\sum_{i=1}^{m} X_i^2 ,$$

multiplied by some power of J.

We now transform from the variables X to new variables U given by the scheme

$$X_i = \sum_{j=1}^{m} \frac{\partial x_i}{\partial u_j} U_j, \qquad (i = 1, 2 \ldots . m).$$

The Jacobian of this transformation is J, and the discriminant of the quadratic form ΣX_i^2 changes from 1 to J^2. Hence the most general invariant is a function of the quantities $d^\lambda f$, and of the general algebraic invariants of the A's and ΣX_i^2, expressed in the new variables.

Let the form

$$\sum_{i=1}^{m} X_i^2 \text{ become } \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} U_i U_j,$$

then it will be shown that the coefficients of the algebraic forms may be expressed in terms of the a's, the f's and their derivatives with respect to the variables u. We proceed to calculate these forms in the new variables.

$$A_1 \text{ becomes } \sum_{i=1}^{m} U_i \frac{\partial f}{\partial u_i}.$$

In the general case

$$A_p^{(\rho)} = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \cdot\cdot \sum_{\lambda=1}^{m} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \cdot\cdot \frac{\partial x_i}{\partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdot\cdot \frac{\partial^p f^{(\rho)}}{\partial x_i \partial x_j \partial x_k \cdot\cdot} U_\lambda U_\mu U_\nu \cdot\cdot,$$

where there are n letters $i, j, k \ldots .$, and also n letters $\lambda, \mu, \nu, \ldots .$ We use

$$\frac{p!}{\lambda!\, \mu!\, \nu! \ldots .}\, {}_pB_{\lambda, \mu, \nu, \ldots .}$$

to denote the coefficient of U_λ, U_μ, $U_\nu \ldots .$ in the above expression for A_p.

It is convenient to introduce at this stage the well known three index symbols of Christoffel

$$\begin{bmatrix} \lambda, \mu \\ \nu \end{bmatrix} \equiv \tfrac{1}{2} \left\{ \frac{\partial a_{\lambda\nu}}{\partial u_\mu} + \frac{\partial a_{\mu\nu}}{\partial u_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial u_\nu} \right\}$$

and we have the relations

$$\sum_{i=1}^{m} \frac{\partial^2 x_i}{\partial u_j\, \partial u_k} \frac{\partial x_i}{\partial u_\lambda} = \begin{bmatrix} j, k \\ \lambda \end{bmatrix}.$$

Now $\dfrac{\partial}{\partial u_\rho}\, {}_qB_{\lambda, \mu, \nu, \ldots} = {}_{q+1}B_{\rho, \lambda, \mu, \nu, \ldots}$

$$+ \sum_{ijk\ldots}^{m} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \cdot \left\{ \sum \frac{\partial^2 x_i}{\partial u_\rho\, \partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots \right\}$$

where the second Σ denotes that there is a term corresponding to each of the letters λ, μ, ν, ... and that these are added.

Also $${}_qB_{\rho, \mu, \nu, \ldots} = \sum_{i, j, k, \ldots}^{m} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \frac{\partial x_i}{\partial u_\rho} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots\cdots$$

We solve the m equations obtained by giving ρ the values 1, 2, ... m, for the quantities

$$\sum_{j, k, \ldots} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots,$$

and substitute the result in the above equation. Hence

$$\frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda, \mu, \nu, \ldots} = {}_{q+1}B_{\rho, \lambda, \mu, \nu, \ldots}$$

$$+ \sum \left\{ \sum_{i=1}^{m} \sum_{p=1}^{m} {}_qB_{p, \mu, \nu, \ldots}\, M_p^i \frac{\partial^2 x_i}{\partial u_\rho\, \partial u_\lambda} \right\} \frac{1}{J},$$

where M_p^i is the cofactor of $\dfrac{\partial x_i}{\partial u_p}$ in J. This equation may be written

$${}_{q+1}B_{\rho, \lambda, \mu, \nu, \ldots} - \frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda, \mu, \nu, \ldots} =$$

$$\Sigma \frac{1}{J} \begin{vmatrix} 0 & \dfrac{\partial^2 x_1}{\partial u_\rho \partial u_\lambda}, & \dfrac{\partial^2 x_2}{\partial u_\rho \partial u_\lambda}, & \cdots & \dfrac{\partial^2 x_m}{\partial u_\rho \partial u_\lambda} \\ {}_qB_{1, \mu, \nu, \ldots} & \dfrac{\partial x_1}{\partial u_1}, & \dfrac{\partial x_2}{\partial u_1}, & \cdots & \dfrac{\partial x_m}{\partial u_1} \\ {}_qB_{2, \mu, \nu, \ldots} & \dfrac{\partial x_1}{\partial u_2}, & \dfrac{\partial x_2}{\partial u_2}, & \cdots & \dfrac{\partial x_m}{\partial u_2} \\ \vdots & & & & \vdots \\ {}_qB_{m, \mu, \nu, \ldots} & \dfrac{\partial u_m}{\partial x_m}, & \cdots & \cdots & \dfrac{\partial x_m}{\partial u_m} \end{vmatrix}$$

We multiply the determinant on the right by J, and observe that $J^2 = \Delta$, where Δ is the discriminant of the quadratic form

$$\sum_{r,s} a_{rs}\, U_r\, U_s\,.$$

We thus obtain the result

$$
{}_{q+1}B_{\rho,\lambda,\mu,\nu,\dots} - \frac{\partial}{\partial u_\rho}\,{}_qB_{\lambda,\mu,\nu,\dots} =
$$

$$
\Sigma \begin{vmatrix}
0 & , \begin{bmatrix}\rho,\lambda\\1\end{bmatrix}, & \begin{bmatrix}\rho,\lambda\\2\end{bmatrix}, & \dots & \begin{bmatrix}\rho,\lambda\\m\end{bmatrix} \\
{}_qB_{1,\mu,\nu,\dots} & a_{11} & a_{12} & \dots & a_{1m} \\
{}_qB_{2,\mu,\nu,\dots} & a_{21} & a_{22} & \dots & a_{2m} \\
\vdots & & \cdot & & \vdots \\
{}_qB_{m,\mu,\nu,\dots} & a_{m1} & a_{m2} & \dots & a_{mm}
\end{vmatrix} \frac{1}{\Delta}
$$

Where the summation contains a term for each letter $\lambda, \mu, \nu, \dots$.

Multiply both sides of this equation by $U_\rho\, U_\lambda\, U_\mu\, U_\nu \dots$ and sum for all values of $\rho, \lambda, \mu, \nu, \dots$. Also let the operator

$$\sum_{\rho=1}^{m} U_\rho \frac{\partial}{\partial u_\rho}\text{, be denoted by } \Omega.$$

Then $S_{q+1} - \Omega\, S_q$

$$
= \frac{1}{\Delta}\begin{vmatrix}
0, & F_1 & , F_2 & , \dots F_m \\
\frac{\partial S_q}{\partial U_1}, & a_{11} & , a_{12} & , \dots a_{1m} \\
\vdots & & & \vdots \\
\frac{\partial S_q}{\partial U_m}, & a_{m1} & , a_{m2} & , \dots a_{mm}
\end{vmatrix} \qquad \dots \qquad (3)
$$

Where S_q denotes the form previously called A_q, and F_p is the quadratic form

$$\sum_{\rho=1}^{m}\sum_{\lambda=1}^{m}\begin{bmatrix}\rho,\lambda\\p\end{bmatrix}U_\rho U_\lambda.$$

It thus appears that the coefficients of S_{q+1} involve the a's, their derivatives, the coefficients of the forms S_q and their derivatives. But S_1 is obviously Ωf, and therefore the coefficients of S_q involve the a's, their derivatives of orders up

to $q-1$, and the derivatives of the f's of orders up to q. These forms may be readily calculated in succession by means of the given equation.

The final result may now be stated. Let $u_1, u_2 \ldots\ldots u_m$ be any coördinates in space of m dimensions, and let the element of length ds be given by the equation

$$ds^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, du_r\, du_s .$$

A complete functionally independent system of differential invariants of orders up to and including n is given by the expressions

$$d^{\lambda} f^{(\rho)} \qquad (\lambda = 1, 2, \ldots\ldots n;\ \rho = 1, 2, \ldots\ldots m)$$

together with the algebraic invariants of certain m-ary forms. These forms consist of:

(I) The quadratic $\qquad \displaystyle\sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, U_r\, U_s .$

(II) n forms of orders $1, 2, \ldots\ldots n$, corresponding to each function $f(u_1, u_2, \ldots\ldots u_m)$.

The coefficients of the forms (II) involve the derivatives of the f's, the a's and their derivatives. Equation (3) enables these forms to be written down in succession.*

§3. We next consider any manifold of dimensions r in the m dimensional space. In this case there are two types of invariants. There is, in the first place, a class of invariants corresponding to transformations which preserve the 'shape' of the manifold, that is to say do not alter distances apart of points on the manifold, the distances being measured through the m dimensional space. These contain a subclass of deformation invariants, namely functions which remain unchanged when the manifold is subjected to a transformation which merely preserves lengths measured in the manifold itself. Let the manifold in question be given by $u_{r+\lambda} = 0$ $(\lambda = 1 \ldots\ldots m-r)$.

---

*Forsyth, in his memoir already quoted, "*The Differential Invariants of Space*" (p. 320), gives three of the forms S, namely, those for a function ϕ when $m=3$ and $n=3$.

Then the first class of invariants is included in the general space set. We may now, however, take as $m - r$ of the functions f the variables $u_{r+\lambda}$. In addition $du_{r+\lambda} = 0 \;\ldots.\; d^n u_{r+\lambda} = 0$.

It is quite easy to see that the result of the last paragraph may be modified by omitting the invariants $df^{(1)} \ldots. df^{(m)}$, and taking $du_1, du_2, \ldots. du_m$, as the variables in the algebraic forms. Then the set of forms includes the $m - r$ linear ones $du_{r+\lambda}$ $(\lambda = 1, 2, \ldots. m - r)$ and therefore the invariants of the manifold are the invariants and covariants of the forms in r variables obtained by putting $du_{r+\lambda} = 0$ in the m-ary set,* and in addition there are the invariants

$$d^p f^{(\rho)} \; (p = 2, \ldots. n;\; \rho = 1, 2 \ldots. m).$$

In all the coefficients of the forms the $u_{r+\lambda}$'s $(\lambda = 1 \ldots. m - r)$ are finally put equal to zero.

There are two types of forms to be calculated, namely those belonging to a function $u_{r+\lambda}$, and those belonging to a function $\phi\,(u_1, u_2, \ldots. u_r)$. We make the assumption that the variables $u_{r+\lambda}$ may be so selected that in the expression

$$ds^2 = \sum_{\alpha\beta}^{m} a_{\alpha\beta}\, du_\alpha\, du_\beta$$

the coefficient $a_{\lambda\mu} = 0$ if $\lambda > r$, $\mu \leq r$. With this selection of variables

$$S_2^{(r+\lambda)} \text{ becomes } \frac{M_\lambda}{M_0},$$

where M_i is the cofactor of b_i in

$$\begin{vmatrix} b_0 & F_{r+1}, & \ldots. & F_m \\ b_1 & a_{r+1,r+1} & \ldots. & a_{r+1,m} \\ \vdots & & & \vdots \\ b_{m-r} & a_{r+1,m} & \ldots. & a_{m,m} \end{vmatrix}$$

The coefficients of these forms obviously cannot be expressed in terms of the magnitudes $a_{\alpha\beta}$ $(\alpha, \beta = 1, 2 \ldots. r)$. Let the coefficients in question be $L_{ij}^{(r+\lambda}$ $(ij = 1 \ldots. r)$.

* See Grace and Young. Algebra of Invariants (1903), p. 266.

Then $S_{q+1}^{(r+\lambda)} - \Omega_1 S_q^{(r+\lambda)}$

$$= \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & F_2 & \dots & F_r \\ \frac{\partial S_q^{(r+\lambda)}}{\partial U_1} & a_{11} & a_{12} & \dots & a_{1r} \\ \vdots & & & & \vdots \\ \frac{\partial S_q^{(r+\lambda)}}{\partial U_r} & a_{r1} & & \dots & a_{rr} \end{vmatrix}$$

where Δ_1 is the discriminant of

$$\sum_{\alpha\beta}^{r} a_{\alpha\beta} U_\alpha U_\beta, \qquad \Omega_1 \equiv \sum_{=1}^{r} U_j \frac{\partial}{\partial u_j},$$

and the variables U_j of the forms are du_j $(j = 1, \dots r)$. Now the coefficients of the forms $F_1, F_2, \dots F_r$ are the three index symbols $\begin{bmatrix} ij \\ k \end{bmatrix}$ $(i, j, k = 1, 2 \dots r)$ and they are therefore expressible in terms of the derivatives of the $a_{\alpha\beta}$'s where $\alpha, \beta \not> r$.

It therefore follows that the coefficients of the forms $S^{(r+\lambda)}$ $(\lambda = 1, 2 \dots m - r)$ may all be expressed in terms of the coefficients of ds^2 for the manifold, their derivatives, and the derivatives of the coefficients of the forms $S_2^{(r+\lambda)}$.

The coefficients of the forms $S_q^{(r+\lambda)}$ are seen to be the generalization of the "Fundamental Magnitudes"* of order q of a surface in space of three dimensions.

The forms corresponding to a function ϕ $(u_1, u_2, \dots u_r)$ are exactly similar to the general forms for space, for $S_1 = \Omega_1 \phi$.

$$S_{q+1} = \Omega_1 S_q + \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & \dots & F_r \\ \frac{\partial S_q}{\partial U_1} & a_{11} & \dots & a_{1r} \\ \vdots & & & \vdots \\ \frac{\partial S_q}{\partial U_r} & a_{r1} & \dots & a_{rr} \end{vmatrix}.$$

* See Forsyth, Messenger of Mathematics, Vol. 32, 1903, pp. 68 et seq.

We may now get rid of covariants of the forms in question by taking account of the invariants $d\phi_1 \ldots\ldots d\phi_r$, and the final solution of problem II is obtained. The general statement is the following: Let there be any manifold in space of m dimensions, and let its linear element be given by the equation

$$ds^2 = \sum_{\alpha, \beta=1}^{r} a_{\alpha\beta}\, du_\alpha\, du_\beta .$$

In addition let there be r functions $\phi\,(u_1 \ldots\ldots u_r)$. Then a complete system of algebraically independent differential invariants of the manifold is given by

$$d^\lambda\, \phi^{(\sigma)} \qquad (\lambda = 1, 2 \ldots\ldots n;\ \sigma = 1, 2 \ldots\ldots r)$$

together with the algebraically independent invariants of a system of r-ary forms. These forms are:

(1) A quadratic $\Sigma a_{\alpha\beta} du_\alpha du_\beta$.

(2) $m - r$ quadratics whose coefficients are the fundamental magnitudes of the second order for the manifold.

(3) $m - r$ forms of order q ($q = 3, \ldots\ldots n$) of which the coefficients are the fundamental magnitudes of order q, and which may be expressed in terms of the a's, their derivatives, and the fundamental magnitudes of the second order and their derivatives.

(4) A set of n forms of orders $1, 2 \ldots\ldots n$ corresponding to each function ϕ. The coefficients are functions of the a's, the ϕ's, and their derivatives. Forsyth* gives a particular case of this solution, namely that for $n = 3$, $m = 3$, $r = 2$. He, however, assumes at the beginning that there exist certain invariant differential forms corresponding to the sets (2) and (3) whereas in our case these forms have arisen naturally in the course of the development of the method.

§4. Problem (III) still remains for consideration. In the first place, it is clear that $d^\lambda\, \phi^{(\sigma)}$ ($\lambda = 1, 2 \ldots\ldots n$; $\sigma = 1, 2 \ldots\ldots r$) and the algebraic invariants of the sets (1) and (4) of algebraic forms given in the preceding section are deformation invariants of the quadratic differential form ds^2.†

* Philosophical Transactions, Ser. A, Vol. 201 (1903), p. 357.

† There is one condition to which the differential form considered is subject. It is assumed possible to express it as the sum of the squares of m perfect differentials. This condition involves no limitation on the form, for it may be proved that any form can be so expressed, provided $m \overline{<} \frac{1}{2} r (r + 1)$. See Goursat, Leçons sur l'intégration des Équations aux Dérivées Partielles du premier ordre (1891), p. 11.

In addition to these there are others due to the sets (2) and (3), for it is easy to see that there are relations among the fundamental magnitudes of the second order, the a's and their derivatives,* and hence some of the invariants due to these sets must be expressible in terms of the derivatives of the a's alone. The solution of problem (I) shows that these are all zero in that particular case. It is interesting in this connection, to note the six quantities $\Theta_1, \Theta_2, \ldots. \Theta_6$ obtained by Forsyth † and originally due to Cayley. ‡ The method hitherto pursued leads us no further in the determination of "Gaussian Invariants," and we must have recourse to another. The problem to be solved is the determination of the invariants of a quadratic form, when there are no associated functions ϕ. Use is made of Lie's general method by the introduction into the group hitherto used of a certain number of new variables which are invariant to the group. ‖ Let the quadratic form be

$$\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk}\, du_h\, du_k,$$

and introduce invariant variables $\alpha_1, \alpha_2, \ldots. \alpha_r$. The a's are functions of the u's, and the u's are taken to be functions of the α's, which are a set of independent variables. We now seek for invariants, under the most general transformation on the u's, which involve as variables

(1) the u's and their derivatives with respect to the α's,
(2) the a's and their derivatives with respect to the u's,
subject to the condition that the quadratic form is invariant.

It is easy to see

(a) that $\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk} \frac{\partial u_h}{\partial \alpha_\lambda} \frac{\partial u_k}{\partial \alpha_\mu}$ $(\lambda, \mu = 1, 2, \ldots. r)$ is an absolute invariant,

(b) that if H is any absolute invariant so also is $\frac{\partial H}{\partial \alpha}$.

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* In the case of a surface in space of three dimensions the Gaussian curvature leads to such a relation.
† Philosophical Transactions, Series A, Vol. 202 (1903), p. 306.
‡ Collected Mathematical Papers, Vol. XII, pp. 12, 13.
‖ Lie, Math. Ann., Vol. XXIV (1884), p. 564, lines 22 et seq.

These two statements (a) and (b) are sufficient to give the clue to the complete set of invariants sought.

Let $T_{\lambda\mu}$ denote the expression (a), and form all possible derivatives up to and including the order $n-1$, of the T's with respect to the α's. These are a complete functionally independent set of invariants. It is easy to prove the independence of this set, for if α_λ is taken to be u_λ, $(\lambda = 1, 2 \ldots. r)$ the T's and their derivatives become the a's and their derivatives, and the a's are arbitrary functions of their arguments. Hence the number of functionally independent invariants obtained is

$$r\,\frac{(r+1)}{2}\,\frac{(n+r-1)!}{(r-1)!\;n!}$$

Now the set of equations for these invariants obtained by the Lie process is precisely that considered by C. N. Haskins,* with certain additional terms due to the presence of derivatives of the u's with respect to the a's. He shows that if $n > 3$ the equations are independent, and therefore it is certain that if $n > 3$ our equations are independent. Now they possess altogether as variables

$$r\,\frac{(r+1)}{2}\,\frac{(n+r-1)!}{(r-1)!\;n!}\; a\text{'s and their derivatives,}$$

$$r\left\{\frac{(n+r)!}{(r-1)!\;(n+1)!}-1\right\}\text{ derivatives of the } u\text{'s.}$$

Also the equations are in number

$$r\left\{\frac{(n+r)!}{(r-1)!\;(n+1)!}-1\right\},$$

and therefore they possess exactly

$$\frac{r\,(r+1)}{2}\cdot\frac{(n+r-1)!}{(r-1)!\;n!}$$

functionally independent common solutions. But this number has been obtained, and therefore the system of equations has been completely solved. Any invariant is therefore a function of the T's and their derivatives, and the Gaussian Invari-

* Trans. Amer. Math. Soc., Vol. 3 (1902), pp. 74-76.

ants are the particular functions which do not involve the derivatives of the u's with respect to the α's. These may be obtained by processes of elimination, and hence the problem of the determination of the Gaussian Invariants of a quadratic form is completely solved.

It is clear that the introduction of any functions $\phi\ (u_1, u_2, \ldots u_r)$ leads to additional invariants $\frac{d\phi}{d\alpha}$ etc., and therefore the parameters may be obtained in this way. The extension to any number of forms of any degree is immediate, but this question is reserved for future discussion.

BRYN MAWR COLLEGE, PENNA., *April*, 1905.

III. Regions, Orientation of Curves, Normals, and Related Topics.

10. *Change of Parameter.* The following theorem is stated without proof:

THEOREM. *Let a given simple curve be defined by two sets of equations*

$$\begin{cases} x = \phi(t), \\ y = \psi(t), \end{cases} \quad t_0 \overline{\overline{<}} t \overline{\overline{<}} t_1, \quad \text{and} \quad \begin{cases} x = \bar{\phi}(t), \\ y = \bar{\psi}(t), \end{cases} \quad t_0' \leqq t' \leqq t_1',$$

where ϕ, ψ, $\bar{\phi}$ *and* $\bar{\psi}$ *are single valued, continuous functions.*

(a) *In the case of the open curve, if no two values of* $t\,[t']$ *yield the same point, and if the values of* t *and* t' *which yield the same point of the curve are assigned to each other, then* t *is a single valued, continuous function* $f(t')$, *monotonic and never constant throughout the interval of definition; and the same two points are given as the end points in each case;*

(b) *In the case of the closed curve, if no two values of* $t\,[t']$ *yield the same point unless they differ by a period of the pair of functions, the values of* t *and* t' *which yield the same point of the curve can be assigned to each other in such a way that* $t' = f(t)$, *where* $f(t)$ *is single valued and continuous for all values of* t, *monotonic and never constant.*

The totality of transformations $t' = f(t)$ thus defined form a group G. Such a transformation is said to be *even* if an increase in t yields an increase in t'. The even transformations of G form a subgroup G^+ of G. Any transformation of G is an even transformation or is equivalent to an even transformation followed by the transformation $t' = -t$. The order n of a point with respect to the curve is invariant of any even transformation. If t is replaced by $-t'$ the sign of the order of a point is reversed. Then n^2 is invariant of any transformation of G.

11. *Interior and Exterior.*

THEOREM. *All sufficiently distant points are of order zero with respect to a given closed curve.*

Proof. Let P_1 (x_1, y_1) be a distant point, and let P (x, y) be a variable point on the curve. Let θ be the angle P_1P makes with the positive x-axis. Then by Art. 5,

$$\cos\theta = (x - x_1)/P_1P, \qquad \sin\theta = (y - y_1)/P_1P.$$

Then if $\sqrt{x_1^2 + y_1^2}$ is taken sufficiently large either $\cos\theta$ or $\sin\theta$ never changes its sign as P varies. In either case the maximum variation of θ is less than π. Hence the order of P_1 is zero.

If the points of a continuum are all of order n the *continuum* is defined to be of *order* n. The *exterior* of a simple closed curve is defined to be that one of the two continua into which the curve divides the plane which contains all sufficiently distant points. The other continuum is defined to be the *interior.* It follows that the exterior is of order zero, and the interior of order ± 1. If the interior is of order -1, the parameter can be so chosen that the order of the interior will be $+1$. The *neighborhood of a curve* is a continuum containing all points of the curve, and such that if P is a point of the continuum, and P_1 a suitably chosen point of the curve, then $PP_1 < h$, where h is a positive constant previously chosen as small as either party to a discussion wishes.

We have proved incidentally the following theorem, which for greater clearness we state somewhat freely in geometric language.

THEOREM. *Let P be a variable point on a simple closed regular curve, and A any fixed point not on the curve. Then when P traces the curve and returns to its initial position, the angle which AP makes with the positive x-axis, varying continuously returns to its initial value if A is an exterior point of the curve, and is changed by 2π if A is an interior point.*

12. *Orientation of Curves.* The conception of an *oriented* curve is a generalization of that of a vector. It is often desirable to distinguish the positive from the negative sense along a curve. The process or the result of making this distinction we will call *orientation.* More explicitly, we define an *oriented curve* and then define the *positive sense along* such a curve. An *oriented* simple curve is defined to be an object determined by the two following phenomena:

(a) A simple curve;

(b) One of the two possible permutations AB or BA of the end points of any one open arc of the curve.

If the orientation of a given simple curve is defined by the permutation P_1P_2 of the end points $P_1(t_1)$ and $P_2(t_2)$ of a definite open arc, and $P(t)$ is any point of that arc, then we will agree to choose the parameter so that $t_1 \leqq t \leqq t_2$, and conversely. If a change of parameter is necessary to effect this it can always be accomplished by the transformation $t = -t'$. Thus a permutation of the end

points of any open arc is uniquely determined. Hence a simple curve can be oriented in two and only two distinct ways, and one of these is fully determined by a permutation of the end points of an arbitrary arc. With this agreement as to the choice of parameter, a point is said to *trace* a simple curve in the *positive sense* if its parameter *increases* continuously. A line integral is said to be *extended along* the arc $t_1\, t_2$ in the *positive sense* if t_1 is the lower limit and t_2 the upper limit of integration. The sign of the order of a point with respect to a curve is reversed by reversing the orientation of the curve.

In general the orientation of an open curve is entirely arbitrary. If a simple curve is considered as part or all of the boundary of a definite region, we will agree that the curve shall be so oriented that the region shall be of order one greater than the region from which the curve separates it (see Fig. 5). If a

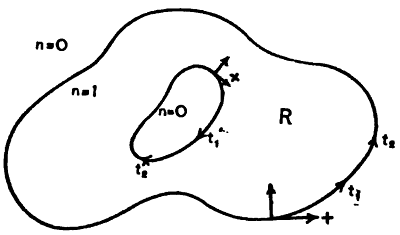

Fig. 5

closed curve is not explicitly considered as a boundary of a region exterior to it, it shall be oriented so that its interior is of order one greater than its exterior, in other words, so that if O is an interior point and P traces the curve in the positive sense returning to its initial position, the angle which OP makes with a fixed line varying continuously shall be *increased* by 2π (see Art. 11).

If a one-to-one relation is established between the points of two simple curves and the parameters have been chosen as above, then they are said to have the *same orientation*, or to be *similarly oriented* if the parameter of one is an increasing function of that of the other. They are said to have *opposite orientations*, or to be *oppositely oriented* if the parameter of one is a decreasing function of that of the other. A special case of this is that in which two curves coincide along a given arc. In this case coincident points in the two curves are assigned to each other, unless otherwise specified.

THEOREM. *Let two plane regions R_1 and R_2 each form the interior of a simple closed curve C_1 and C_2 respectively, satisfying Condition A. Let a segment σ_1 of C_1 coincide with a segment σ_2 of C_2, then*

(a) *If R_1 and R_2 are exterior to each other, the orientation of σ_1 is opposite to that of σ_2;*

(b) *If R_1 is wholly interior to R_2, the orientation of σ_1 is the same as that of σ_2.*

Proof (a). Choose a part or all of σ_1 (or σ_2) which can be represented in the form

$$(1) \quad y = f(x), \qquad \text{or else in the form} \qquad (2) \quad x = f(y),$$

where f is a single valued, continuous function. Let P_0 be any point of this segment not an end point. The orientation of σ_1 can not be the same as that of σ_2, for suppose it were. Then by Art. 9, First Lemma, near P_0 there are two points B and B' such that the order of B with respect to either curve is greater than that of B' by unity. Hence by Art. 11 B is interior to each curve, which is contrary to hypothesis. The second case is proved similarly.

The property of the plane stated in this theorem is later taken as the definition of a bilateral surface. It is not true on a unilateral surface. Thus it will follow that the plane is bilateral. Goursat tacitly assumes this theorem or an equivalent one in his proof of Cauchy's Integral Theorem.* It is assumed whenever an integral taken around any region R is assumed to be equal to the sum of the integrals taken around the mutually exclusive regions of which R consists, and in analogous cases involving variation, or analytic continuation along closed paths in the study of multiple valued functions.

13. *Tangents and Normals.* If a simple closed regular curve is represented by the equations

$$x = \phi(t), \qquad y = \psi(t)$$

and if its orientation has been defined, and the parameter has been chosen according to the specifications of Art. 12, then the *positive tangent* at a point $P_0 \ (t_0)$ at which the curve is smooth is the vector $P_0 \ (s_0)\, P_1 \ (s_1)$ defined by the equations

$$\begin{aligned} x &= s\phi'(t_0) + \phi(t_0), \\ y &= s\psi'(t_0) + \psi(t_0), \end{aligned} \qquad (s_0 = 0 \leqq s \leqq s_1).$$

* Acta Mathematica, *t.* IV, p. 197. Or see Harkness and Morley, *Theory of Functions* (1893), p. 164

A *normal* to the curve at a point $P_0 \quad (t_0)$ at which the curve is smooth is a vector defined by the equations

$$x = -\varepsilon s\phi'(t_0) + \phi(t_0), \qquad (s_0 = 0 \leqq s \leqq s_1),$$
$$y = \varepsilon s\psi'(t_0) + \psi(t_0), \qquad (\varepsilon = \pm 1).$$

If this enters the interior of the curve in the neighborhood of the curve it is called the *inner normal.* It follows from the proof of the first lemma, Art. 9, that in the case of the inner normal $\varepsilon = +1$, and the inner normal makes an angle of $+\pi/2$ with the positive tangent. Even when the region is exterior to one of its boundaries, these conclusions are equally true of that normal which enters the interior of the region provided the orientation of the boundary is chosen according to the specifications of Art. 12 (see Fig. 5).

14. *Regions and Boundaries.* As an illustration of a class of theorems which are often assumed without even mention, but which are by no means trivial, the following theorems are stated, mostly without proof. A *loop-cut* is defined to be a simple closed curve lying wholly in a continuum under consideration.

THEOREM I. *The totality R^- of the points of a plane continuum R not on a given loop-cut L satisfying Condition A form two continua, one of which is wholly interior and one wholly exterior to the loop-cut, and every point of the loop cut is a boundary point of each.*

The proof is similar to that of the main theorem of Art. 9, which is a special case of this.

THEOREM II. *If a loop-cut L is drawn in the interior of a closed simple curve C, each satisfying Condition A:*

(a) *The interior of L lies wholly interior to C, and is wholly bounded by L;*

(b) *The exterior and perimeter of C lies wholly exterior to L, and the exterior of C is bounded wholly by C;*

(c) *There exist points exterior to L and interior to C, and they form a continuum of which C and L form the total boundary.*

Proof. The main points of the proof may be exhibited in outline as follows:

Each point of the plane belongs to one of nine mutually exclusive classes:

	C_i	C	C_e
L_i		*No point.* (3)	*No point.* (2)
L		*No point.* (1)	*No point.* (1)
L_e			

where C_i and C_e denote the interior and exterior respectively of C, etc.

(1) By hypothesis, no point of L belongs to C or to C_e.

(2) No point is in C_e and L_i. Suppose P were in C_e and L_i. Choose a distant point A. This lies in C_e and in L_e. Hence A and P can be joined by a curve wholly in C_e. This curve must cut L. Hence a point of L is in C_e. This contradicts (1).

(3) Suppose a point P of C were in L_i, then all points near P are in L_i. But there are points of C_e near P. This contradicts (2).

By Theorem I the points of C_i consist of the curve L, and two continua belonging to L_i and L_e respectively. From the diagram it is seen that C is wholly in L_e. Hence by Theorem I the points of L_e consist of the curve C and two continua belonging to C_i and C_e respectively. Hence there are points of each class left blank in the diagram. The first clause of (a), (b), and (c) can now be read off from the diagram. The remainder of the proof is left to the reader.

THEOREM III. *If two simple closed curves C and L each satisfying Condition A are wholly exterior to each other:*

(a) *The interior of C is wholly exterior to L, and is wholly bounded by C, and similarly interchanging letters;*

(b) *There exist points exterior to each, and these form a continuum bounded wholly by C and L.*

The proof is similar to that of Theorem II.

THEOREM IV. *If n simple closed curves satisfying Condition A have no point in common:*

(*a*) *A necessary and sufficient condition that they form the total boundary of an infinite region is that they lie wholly exterior to each other; the region so bounded is exterior to each;*

(*b*) *A necessary and sufficient condition that they form the total boundary of a finite region is that $n - 1$ of the curves are wholly interior to the remaining one and wholly exterior to each other; the region so bounded is exterior to each of the $n - 1$ curves and interior to the remaining one.*

This can be proved by mathematical induction.

PART II.

IN THREE DIMENSIONAL SPACE.

I.—FUNDAMENTAL CONCEPTIONS.

15. Some of the fundamental conceptions made use of in the following chapters have been discussed in the introduction for space of two dimensions. Those definitions and principles will now be extended to space of three dimensions by the addition of a third variable, without further comment, whenever no difficulty presents itself in so doing. We shall prove the theorem that a simple closed surface divides space into two continua, first for a very restricted class of surfaces, and later indicate how the proof can be extended to more general cases. The proof follows a method similar to that used in two dimensions. A preliminary discussion of certain fundamental conceptions is necessary.

16. *Surfaces.* *A smooth simple closed surface* is an assemblage of points $P(x, y, z)$ defined as follows:

(*a*) If $P_0(x_0, y_0, z_0)$ is a point of the assemblage, it is possible to choose three equations

$$x = \phi(u, v), \qquad y = \psi(u, v), \qquad z = \chi(u, v), \tag{A}$$

where

$$x_0 = \phi(u_0, v_0), \quad y_0 = \psi(u_0, v_0), \quad z_0 = \chi(u_0, v_0),$$

where ϕ, ψ, χ are single valued near (u_0, v_0), where all points given by these equations near (u_0, v_0) are points of the assemblage, and where no point (x, y, z) is given by two distinct points (u, v) near (u_0, v_0).

(*b*) The assemblage is *simple*, that is, if P_0 (x_0, y_0, z_0) is a point of the assemblage, all points in the three dimensional neighborhood of P_0 can be given by one set of parametric equations (A) as just defined.

(*c*) The assemblage is *complete*, that is, if $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ is a limiting point of the assemblage, then it shall belong to the assemblage.

(*d*) The assemblage is *connected*, that is, if $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are any two points of the assemblage, then it is possible to draw a simple curve

$$x = \lambda(t), \qquad y = \mu(t), \qquad z = \nu(t), \qquad\qquad (t_0 \leqq t \leqq t_1),$$

having P_0 and P_1 as end points and such that all points of the curve are points of the assemblage.

(*e*) The assemblage lies in a *finite region*, that is, it is possible to choose a constant G so that if $P(x, y, z)$ is a point of the assemblage, then

$$|x| + |y| + |z| < G.$$

(*f*) The assemblage is *smooth at every point*, or simply *smooth*, that is, if P_0 (x_0, y_0, z_0) is a point of the assemblage, it is possible to choose the equations (A) so that the first partial derivatives

$$\phi_u \left(= \frac{\partial \phi}{\partial u}\right), \phi_v, \psi_u, \psi_v, \chi_u, \chi_v$$

are single valued and continuous near $(u_0, v_0,)$, and so that the Jacobians

$$J_x = \begin{vmatrix} \psi_u & \chi_u \\ \psi_v & \chi_v \end{vmatrix}, \quad J_y = \begin{vmatrix} \psi_u & \phi_u \\ \chi_v & \phi_v \end{vmatrix}, \quad J_z = \begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}$$

do not all vanish at P_0.

By virtue of (*a*), (*c*) and (*e*) the assemblage is said to be *closed*.

17. *Dissection of Surfaces.*

THEOREM I. *A smooth simple closed surface can be divided into a finite number of parts, each of which has the following properties:*

(*a*) *It can be represented by an equation of one of the following forms:*

$$x = f(y, z), \quad \text{or} \quad y = f(z, x), \quad \text{or} \quad z = f(x, y),$$

where f and its first partial derivatives are single-valued and continuous;

(*b*) *It is bounded by one simple regular curve;*

(*c*) *It can be included in a sphere of arbitrarily preassigned radius.*

To prove this we divide space into cubes of edge δ. If δ is chosen sufficiently small, either the part of the surface in any cube consists of a finite number of prices each of which satisfies the requirements of the theorem, or, in case the surface is tangent to a face, the part of the surface in the two cubes having this face in common satisfies the requirements of the theorem. The details of the arithmetization are omitted.

Most of the discussions which follow apply to any simple surface which can be dissected as specified in Theorem I. We shall describe such a surface by saying that it satisfies the following condition:

Condition B: A surface is said to satisfy *Condition B* if it consists of a finite number of parts each of which answers to the description in Theorem I, and if, moreover, the surface satisfies conditions (b, c, d, e) of Art. 15. If it also satisfies (a) it is said to be *closed.*

18. *Parametric Representation of Surfaces.* The following theorems relate to the possibility of representing a given surface by two different systems of parametric equations, and to the relations of the two parametric planes. The transformations involved are analogous to the transformation $t = f(t')$ by which the parameter t of a simple curve is replaced by a different parameter t', where $f(t')$ is monotonic and never constant (Art. 10). The theorems are given without proof.

THEOREM II. *If two planes R and R' are mapped in a one to one and continuous manner on each other, and a simple closed curve C in R is mapped on a simple closed curve C' in R', each of which is oriented, then*

(*a*) *Any interior (exterior) point of C is mapped on an interior (exterior) point of C';*

(*b*) *All such transformations form a group G; of these there are transformations, called* EVEN *transformations, for which the curves of every such pair have the same orientation, and these transformations form a subgroup G^+ of G; there are transformations called* ODD *transformations for which the curves of every such pair have opposite orientations, in particular the transformation $x' = x$, $y' = -y$; the totality of even and odd transformations exhaust G; moreover, the group of odd transformations can be generated by the particular odd transformation $x' = x$, $y' = -y$ in succession with each of the even transformations;*

(*c*) *If the Jacobian of such a transformation is defined and continuous at every point of the regions involved, and does not vanish in them, then the necessary and sufficient condition that the transformation is even is that the Jacobian is positive at every point.*

THEOREM III. *If a finite simple surface region R including its boundary, which is a simple closed curve C, is mapped in a one to one and continuous manner on a portion R' of a plane, then R' forms the interior and boundary of the closed curve C' on which C is mapped.*

19. *Unilateral and Bilateral Surfaces. Orientation.* Given any simple surface. Let R_i be any complete open region of the surface, and let C_i be a simple closed curve forming part or all of its boundary. If, having oriented one such boundary, it is then possible in one and only one way to orient every such boundary so that if C_i and C_j are any two of these having a common segment σ,

(*a*) C_i and C_j shall be oppositely oriented along σ when R_i and R_j are exterior to each other, and

(*b*) C_i and C_j shall be similarly oriented when either R_i or R_j is interior to the other,

then the surface is said to be *bilateral*. If this is not possible the surface is said to be *unilateral*. We will agree that if one such curve on a bilateral surface is oriented, then the orientation of every such curve on the surface shall be consistent with the above specifications.

Let us now extend to surfaces the idea of orientation. We define an *oriented simple surface* to be an object determined by the three following phenomena:

(*a*) A simple bilateral surface;

(*b*) A definite complete open region of the surface;

(*c*) An oriented simple curve forming part or all of the boundary of that region.

It follows that a simple bilateral surface can be oriented in two and only two distinct ways. If a simple bilateral surface is oriented and is represented by three equations of the form

$$x = \phi(u, v), \qquad y = \psi(u, v), \qquad z = \chi(u, v),$$

and if a complete open region R_i of the surface, bounded by C_i, corresponds to the region R_i' bounded by C_i' of the uv-plane, then if C_i and C_i' are not similarly oriented they will be after the substitution $u = -u'$, $v = v'$. We shall assume that the parameter has been so chosen. Then by the theorem of Art. 12 and Theorem III of Art. 18 the same is true for every such curve. Thus the different parts of the surface may be given by different analytic representations and the validity and definiteness of our definitions be not affected.

II. Solid Angles and Order of a Point.

20. *Solid Angles.* Let us start from the ordinary conception of a solid angle. Define a system of spherical coordinates by the relations

$$\begin{aligned} x &= \rho \sin \phi \cos \theta, \\ y &= \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi. \end{aligned}$$

We shall speak of the line $\phi = 0$ and $\phi = \pi$ as the *polar axis*. Let a piece R of a surface be defined by the equation

$$\rho = f(\theta, \phi),$$

where $f(\theta, \phi)$ is a single valued function of θ and ϕ throughout a region R' of the surface of the unit sphere. Then, according to the ordinary conception, the solid angle subtended by R at the origin is the area of R', which is given by the integral

$$\int\int_{R'} \sin \phi \, d\theta \, d\phi.$$

We wish to extend this definition in such a way that the solid angle shall be susceptible of an algebraic sign. To illustrate our purpose, consider the convex surface of a circular cylinder so situated that the origin is exterior to it, and so that it is not pierced by the polar axis, and so that some of the radii vectores cut it in two points. It can be divided into two parts so that each can be represented by one equation of the form $\rho = f(\theta, \phi)$. We wish to define the solid angle subtended by the cylindrical surface so that the contribution of one of these parts shall be positive, and the other negative, and so that the total solid angle shall be the algebraic sum of the solid angles subtended by these two parts. If this surface is represented parametrically by the equations

$$\rho = \mathrm{P}(u, v), \qquad \theta = \Theta(u, v), \qquad \phi = \Phi(u, v),$$

under suitable restrictions as to continuity, it will be observed that the Jacobian

$$\frac{D(\theta, \phi)}{D(u, v)}$$

will be positive throughout one of these parts of the cylinder, and negative throughout the other. If now in the double integral above we replace θ, ϕ by the new variables u, v we obtain the integral

$$\int\int \sin \phi \frac{D(\theta, \phi)}{D(u, v)} du \, dv,*$$

extended over the total cylindrical surface. In this form the Jacobian takes care of the sign, and thus yields in one integral the result desired. Guided by this illustration we shall proceed to formulate a general definition of a solid angle.

Given any oriented bilateral surface R referred to a system of rectangular coordinates, and represented by one or more sets of equations of the form

$$x = X(u, v), \qquad y = Y(u, v) \qquad z = Z(u, v),$$

where the parameters are so chosen as to satisfy the requirements of Art. 19.

* See, for example, Goursat, *Cours d'Analyse*, Vol. I, §128.

Let $O(x_0, y_0, z_0)$ be any fixed point not on the surface. Change to a new system of rectangular axes with O as origin by a transformation having a positive determinant. Then change to a system of spherical coordinates having O as origin and defined by the equations given above. R can now be represented by one or more sets of equations of the form

$$\rho = \mathrm{P}(u, v), \qquad \phi = \Phi(u, v), \qquad \theta = \Theta(u, v).$$

We shall at first require that the surface shall not be pierced by the polar axis. Later we shall remove this restriction. We define the *solid angle* subtended by R at O to be the integral

$$\int\int_{R'} \sin\phi \frac{D(\theta, \phi)}{D(u, v)} du\, dv, \tag{A}$$

where R' is the totality of the regions in the uv-planes corresponding to R, and where $du\, dv$ is essentially positive. It follows that the solid angle is invariant of any change of parameter made by a transformation having a positive Jacobian at every point. In particular it is immaterial whether the surface is represented by one or many sets of parametric equations.

If the surface has a point $P_1(x_1, y_1, z_1, \rho_1)$ on the polar axis, the integrand is not defined at that point as the Jacobian may become infinite. Let a point P approach P_1. Then it can be shown that

$$\lim \sin\phi \frac{D(\theta, \phi)}{D(u, v)} = \frac{1}{\rho_1^2} \frac{D(x_1, y_1)}{D(u, v)},$$

which is finite and defined. The integrand at such a point shall now be defined to be that limit. Then the integral is a fully defined proper integral. We now define the solid angle in all cases to be the integral (A).

Any rotation of the system of spherical axes about O is accomplished by introducing θ' and ϕ' in place of θ and ϕ by a transformation having a positive Jacobian. By simply making this substitution it can be shown that the integrand, and hence the solid angle, is invariant of this transformation. It follows that the solid angle is independent of the particular choice of polar axis, and is invariant of any change in the original system of rectangular axes made by a transformation with positive Jacobian.

21. *Solid Angle in Terms of a Line Integral.* We shall now express the solid angle by means of the line integral $\int \cos\phi\, d\theta$ taken around the boundary of the

surface. The possibility of doing this is suggested by analogy with Green's theorem in the plane. In fact, in the simpler cases this can be accomplished by a direct application of Green's theorem. From Art. 19 it follows that, if the surface is divided into parts, this integral extended along the entire boundary of the surface is equal to the sum of the integrals of the same function extended along the boundaries of the parts, taken in the positive sense of the curve in each case. This follows since along the common boundary of any two of the parts the integral is extended once in one sense and once in the opposite, and these two integrals cancel each other. We need the following theorem:

THEOREM. *Given any smooth surface, and any point O not on the surface; then it is possible to draw through O a straight line not tangent to the surface, making an arbitrarily small angle with a given line.*

Proof. The condition that a line through O is tangent to the surface, represented by spherical coordinates, is that $D(\theta, \phi)/D(u, v) = 0$ at the point of contact, excluding from consideration points of the surface in the neighborhood of the polar axis. This will be no limitation on the generality of the proof, as the polar axis may be changed if desired. Now divide the surface into a finite number of parts each of which can be represented by an equation of at least one of the following forms:

$$(1)\ \rho = \lambda(\theta, \phi), \quad \text{or} \quad (2)\ \theta = \mu(\phi, \rho), \quad \text{or} \quad (3)\ \phi = \nu(\rho, \theta),$$

where λ, μ, ν, and their first derivatives are single valued and continuous. That this is possible follows directly from Art. 17. The condition that a part can be represented in the first form is that $D(\theta, \phi)/D(u,v) \neq 0$ in the part. Hence no line through O can be tangent at a point of a part of the first class. Discard all parts which can be represented in the first form. Then if the parts are taken sufficiently small, $D(\theta, \phi)/D(u, v) < \varepsilon$ throughout the remainder of the surface, which we will denote by S^-, where ε is an arbitrarily preassigned positive number. Then if α is the solid angle subtended by S^- at O,

$$\begin{aligned} \alpha = & \int\!\!\int_{S^-} \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \\ \leq & \int\!\!\int_{S^-} \left| \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \right| \\ \leq & \int\!\!\int_{S^-} \left| \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \right| \\ \leq & \, \varepsilon \int\!\!\int_{S^-} du\, dv \quad \leq \varepsilon \int\!\!\int_{S} du\, dv = \varepsilon K, \end{aligned}$$

where K is independent of the method of division. Hence by taking the parts sufficiently small the solid angle subtended by S^- can be made less than that subtended by a preassigned region of the surface of a sphere with O as center. Then it is possible to draw a straight line through O piercing this region and not touching S^-. This proves the theorem.

Such a line meets the surface in at most a finite number of points. Let the polar axis be so chosen. Now divide the surface into a finite number of arbitrarily small parts by Art. 17. At most a definite number independent of the size of the parts, contain points on the polar axis. If the parts are taken sufficiently small the contribution of these parts to the solid angle can be made arbitrarily small. Each of the remaining parts can be represented in at least one of the three following forms:

$$(1)\ \rho = f(\theta, \phi), \qquad (2)\ \phi = f(\rho, \theta), \qquad (3)\ \theta = f(\phi, \rho).$$

We shall show first that

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv = -\int \cos\phi\, d\theta \tag{B}$$

in each part which can be represented in the first form. The line integral is to be extended around the boundary of each part in the positive sense. The condition that a part can be so represented is that $D(\theta, \phi)/D(u, v) \neq 0$, and hence has a constant sign in the part. Hence

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv = \int\int \sin\phi\, d\theta\, d\phi, \tag{C}$$

where $d\theta\, d\phi$ has a constant sign, the same as $D(\theta, \phi)/D(u, v)$. We may think of the transformation $\theta = \theta(u, v)$, $\phi = \phi(u, v)$, as a transformation from the uv plane to the $\theta\phi$ plane. Suppose $D(\theta, \phi)/D(u, v) < 0$, and hence $d\theta\, d\phi < 0$. Apply Green's theorem. We obtain

$$\int\int \sin\phi\, d\theta\, d\phi = \int \cos\phi\, d\theta$$

extended along the boundary of the region in the $\theta\phi$ plane in the positive sense, or

$$-\int \cos\phi\, d\theta$$

extended in the negative sense. Since $D(\theta, \phi)/D(u, v) < 0$, by Art. 18, Th. II c, this corresponds to the positive sense of the curve in the uv plane, and by Art. 19, this corresponds to the positive sense of the boundary of the region on the surface. If $D(\theta, \phi)/D(u, v) > 0$ similar reasoning leads to the same result. Hence the solid angle is given by the integral

$$-\int \cos\phi \, d\theta$$

taken in the positive sense, for every region of the first class.

Consider those parts which can be represented in the form $\phi = f(\rho, \theta)$. Then

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)} \, du \, dv = \int\int \sin\phi \frac{\partial\phi}{\partial\rho} \, d\rho \, d\theta .$$

By reasoning similar to that just used the same result is obtained.

We shall adopt a different method for the third case. Let R be any one of the regions already discussed, and C its boundary. Let it be referred to any two systems of spherical coordinates (ρ, θ, ϕ) and (ρ, θ', ϕ') having the same origin and such that no point of R or C is a point of either polar axis. If R is sufficiently small it is possible to construct a conical surface having C as directrix and a point V as vertex so chosen that no element intersects either polar axis. Consider that part of this surface which lies between V and C. Let this be divided into arbitrarily small regions. Then each of these regions can be represented by an equation of at least one of the following forms:

$$\rho = f(\theta, \phi), \qquad \text{or} \qquad \phi = f(\rho, \theta),$$

and also by an equation of at least one of the following forms:

$$\rho = f_1(\theta', \phi'), \qquad \text{or} \qquad \phi' = f_1(\rho, \theta').$$

Hence by each system of coordinates the solid angle is equal to the line integral along C. But the solid angle is invariant of the system of axes. Hence

$$\int_C \cos\phi \, d\theta = \int_C \cos\phi' \, d\theta'.$$

Hence in computing the value of $\int \cos\phi \, d\theta$ around the boundaries of all the

regions of the given surface we may choose the axes arbitrarily for each region. But the axes can always be chosen so a given small region can be represented in one of the forms

$$\rho = f(\theta, \phi) \qquad \text{or} \qquad \phi = f(\rho, \theta).$$

Hence for any sufficiently small region not containing a point of the polar axis the solid angle equals $-\int \cos\phi \, d\theta$ extended along its boundary in the positive sense.

22. *Order of a Point.* Now suppose that the surface is closed. We shall show that the solid angle is a multiple, positive, negative or zero, of 4π. Let the curves q_i be the boundaries of those parts, finite in number, and each arbitrarily small, which have a point in common with the polar axis. These parts may be so chosen that the point in common with the polar axis is an interior point of the part. Let C^- be the total boundaries of the remaining parts. Then since the integral is extended along every boundary once in one sense and once in the opposite,

$$\int_{C^-} \cos\phi \, d\theta + \sum \int_{q_i} \cos\phi \, d\theta = 0.$$

But by taking the parts sufficiently small

$$-\int_{C^-} \cos\phi \, d\theta$$

and hence its equal

$$\sum \int_{q_i} \cos\phi \, d\theta$$

can be made arbitrarily near to the solid angle. But at the same time the latter sum can be made arbitrarily near to

$$\sum \eta_i \int_{q_i} d\theta$$

where $\eta_i = \pm 1$. Hence the solid angle equals

$$\sum \eta_i \int_{q_i} d\theta.$$

On the other hand

$$\int_{C^-} d\theta + \sum \int_{q_i} d\theta = 0,$$

since the integral is extended along every segment once in one sense and once in the opposite. But

$$\int_{C^-} d\theta = 0.$$

This may be shown by taking the parts sufficiently small, and showing that the total variation of θ in any one part is less than 2π, and hence equal to zero. Hence

$$\sum \int_{q_i} d\theta = 0.$$

But

$$\int_{q_i} d\theta = 2 n_i \pi,$$

where n_i is an integer, positive, negative or zero, and the solid angle $\sum \eta_i \int_{q_i} d\theta$ differs from $\sum \int_{q_i} d\theta$ by twice the contribution of those integrals for which η_i is negative. Hence *the solid angle equals* $4 n \pi$, *where* n *is an integer, positive, negative or zero.* Then define the *order* of the point O with respect to the surface to be the number n. The order of a point on the surface is not defined.

The solid angle is invariant of the particular choice of the polar axes, and of any change of the parameters u, v in any part of the surface, provided the transformation has a positive Jacobian. If the parameters are changed at all points of the surface by transformations having negative Jacobians, the sign of the solid angle is changed but its absolute value is invariant. Hence the same statements are true of the order of a point. For a like reason the order of a point is invariant of any change in the rectangular axes to which the surface is referred, effected by a transformation with positive Jacobian.

THEOREM I. *If the order of a point* O *not on the surface is* n, *then all points in the neighborhood of* O *are of order* n.

Proof. The integral defining the solid angle is the integral over a finite region of a uniformly continuous function of all the variables involved, including the coordinates of O. Hence the solid angle, and therefore the order of O is continuous when O is not on the surface. But the order can vary only by a multiple of unity. Hence it is constant.

Corollary. *The points of the order of a given point form one or more continua.*

THEOREM II. *If two points are of different orders with respect to a given surface, then any simple curve joining them has a point in common with the surface.*

The proof is the same as in two dimensions (Art. 8).

III. THE DIVISION OF SPACE BY A CLOSED SURFACE.

23. The theorem that a closed bilateral surface which satisfies Condition B (Art. 17) divides space into two continua is proved by the aid of two lemmas entirely analogous to those used in two dimensions.

FIRST LEMMA. *If P_0 is a point of a closed bilateral surface which satisfies Condition B, then near P_0 there are two points whose orders with respect to the given surface differ by unity.*

Proof. Let the surface be arbitrarily oriented. If the surface is not smooth at P_0, there is a point of the surface near P_0 at which it is smooth, and which may be used instead of P_0. Hence we may assume that the surface is smooth at P_0. Transform to a new set of rectangular coordinates with origin at P_0, and so chosen that the surface near P_0 can be represented by one equation $z = f(x, y)$, where f is single valued and continuous. The axes can at the same time be so chosen that the z-axis has only a finite number of points in common with the surface (Art. 21, Theorem). It is possible to choose two points $O^+(o, o, \delta)$ and $O^-(o, o, -\delta)$ where δ is so chosen that no point of the surface except P_0 lies on the segment $O^- O^+$ of the z-axis. Now refer to two systems of spherical coordinates having the origin at O^+ and O^- respectively, and the positive z-axis as positive polar axis. Cut out from the surface small regions, each containing one of the points common to the surface and the polar axis. In each case the sum of the integrals $\int \cos \phi \, d\theta$, extended around these regions in the positive sense, is arbitrarily near to the solid angle subtended by the surface at the origin (Art. 21). The contribution of the part containing P_0 to the angle at O^+ differs from the contribution of the same part to the angle at O^- by a number arbitrarily near to 4π. That of the remaining parts in the two cases differ by an arbitrarily

small number. But the orders of O^+ and O^- can differ only by integers, hence they differ by unity.

SECOND LEMMA. *Given any three dimensional continuum R, and a surface S:*

$$z = f(x, y), \quad \text{or} \quad y = f(z, x), \quad \text{or} \quad x = f(y, z),$$

where f is single valued and continuous:

(*a*) *If R contains all points of S except possibly its boundary points which may lie in the boundary of R, then the totality R^- of points of R not on S form at most two continua;*

(*b*) *If also S has a simple regular boundary one point of which is in R, then R^- forms one continuum.*

Proof. (*a*) Suppose S can be represented by the equation $z = f(x, y)$. The other cases are similar. (See Fig. 6, in which the surface S is represented

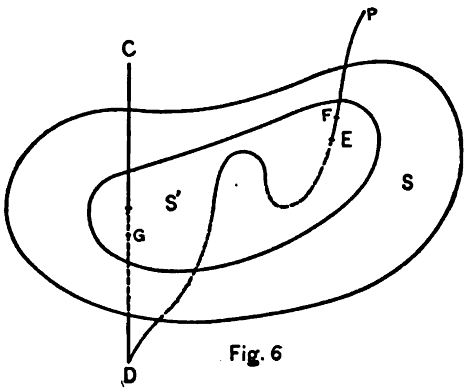

Fig. 6

but not the boundary of R). Draw a straight line CD parallel to the z-axis, lying wholly in R, and bisected at a point of the surface S, and such that C is above the surface S. Let P be any point of R^- which cannot be joined to D by a simple curve wholly in R^-. If there is no such point the theorem is granted. Otherwise join P to D by a simple curve PD wholly in R. This curve will have a point in common with the surface S. Let PE be an arc of PD having one end E on the surface S, but containing no other point of S. Choose a region S' of S whose interior and boundary lies wholly in R, and containing E and the point

common to CD and the surface S. Define two assemblages N^+ and N^- analogous to that of Art. 6, Example 3, as follows:

$$\begin{array}{ll} z = f(x, y) + r, & (x, y) \text{ in } S', \\ 0 < r < h & \text{for } N^+, \\ -h < r < 0 & \text{for } N^-. \end{array}$$

These can be proved to be continua in a manner analogous to that just referred to. Choose a point F on the arc PE, and so near to E that it lies either in N^+ or N^-. Suppose it lay in N^-. Choose a point G on CD in N^-. Then F and G can be joined by a simple curve wholly in N^-. Hence the simple curve $PFGD$ lies wholly in R^-, which is contrary to hypothesis. Hence F must lie in N^+, and by similar reasoning P can be joined to C by a simple curve wholly in R^-. Hence the points of R^- form at most two continua.

(*b*) Suppose the surface is represented as in the first case, but let it be bounded by a simple regular curve having a point P_0 interior to R. If this point is a vertex there is a point of the boundary of S near it which is not a vertex, and which lies in R. Hence we may assume that it is not a vertex. Let the surface now be extended slightly past P_0. By reasoning similar to that of the first case it can be proved that the points of R not on this enlarged surface form at most two continua. If they form one continuum the theorem is granted. If they form two continua, they can be annexed to each other by the adjunction of the points added to the given surface, thus forming one continuum.

MAIN THEOREM. *The points of space not on a given simple closed bilateral surface which satisfies Condition B form two continua, of each of which the entire surface is the total boundary.*

Proof. In the neighborhood of any point of the surface there are two points of different orders with respect to the surface (First Lemma). Hence the points of space not on the surface form *at least two* continua (Art. 22, Th. II). Divide the surface into parts each of which can be represented in at least one of the following forms:

$$x = f(y, z), \quad \text{or} \quad y = f(z, x), \quad \text{or} \quad z = f(x, y).$$

Discard these, one at a time in such an order that each part after the first when discarded shall have a portion at least of its boundary in common with a part

already discarded. Then replace them in reverse order. By the second part of the Second Lemma each of these except the last replaced does not divide the region consisting of all space less the points already cut out. By the first part of the same lemma the last part replaced divides the resulting region into *at most two* continua. Hence the points of space not on the surface form just two continua.

Any point of the surface is a boundary point of each continuum (First Lemma, and Art. 6). Any point not on the surface belongs to one of the continua and hence is not a boundary point. This proves the theorem.

A discussion of interior, exterior and normals might be made analogous to that in two dimensions. More general surfaces, having edges or vertices may be defined in a manner analogous to the definition of a smooth surface given in Art. 16. If such a surface satisfies *Condition B* (Art. 17), then the foregoing discussion applies to it.

On the Definition of Reducible Hypercomplex Number Systems.

II.

BY HEMAN BURR LEONARD.

§1.—*Preliminary.*—The present paper is intended as a completion of the problem studied in a former paper bearing the same title.*

A hypercomplex number system is said to be reducible when, by a proper choice of units, it can be brought to the form

$$E \equiv E_j E_k \equiv e_1 \ldots . e_m e_{m+1} \ldots . e_n,$$

where the following conditions are fulfilled:

A), associativity of E;
C_1), E_j forms a system by itself;
C_2), E_k forms a system by itself;
C_{jk}), $e_j e_k = 0, \quad j = 1, \ldots, m;$
C_{kj}), $e_k e_j = 0, \quad k = m+1, \ldots, n.$

In the former paper were listed seventy-eight different definitions from which the above requirements can be deduced, and these definitions were based on the following twenty conditions:

A_1), $\quad (J_1\ J_2)\ J_3 = J_1\ (J_2\ J_3)$;
A_2), $\quad (K_1 K_2) K_3 = K_1 (K_2 K_3)$;
A_3), $\quad (K_1\ J_1)\ J_2 = K_1\ (J_1\ J_2)$;
A_4), $\quad (J_1\ K_1)\ K_2 = J_1\ (K_1\ K_2)$;
A_5), $\quad (J_1\ K_1)\ J_2 = J_1\ (K_1\ J_2)$;
A_6), $\quad (K_1 J_1)\ K_2 = K_1\ (J_1\ K_2)$;
A_7), $\quad (J_1\ J_2)\ K_1 = J_1\ (J_2\ K_1)$;
A_8), $\quad (K_1\ K_2)\ J_1 = K_1\ (K_2\ J_1)$;
C_1), E_j is closed under multiplication, that is, $J_1\ J_2 = J_3$;
C_2), E_k is closed under multiplication, that is, $K_1\ K_2 = K_3$;

* Epsteen-Leonard, On the Definition of Reducible Hypercomplex Number Systems, American Journal of Mathematics, Vol. XXVII (1905), p. 217.

C^j_{jk}), $\qquad J_1 K_1 = K_2 \qquad (J_2 = 0)$;
C^k_{jk}), $\qquad J_1 K_1 = J_2 \qquad (K_2 = 0)$;
C^j_{kj}), $\qquad K_1 J_1 = K_2 \qquad (J_2 = 0)$;
C^k_{kj}), $\qquad K_1 J_1 = J_2 \qquad (K_2 = 0)$;

C_r), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1 X = 0, \text{ only if } X_1 = 0;$$

C_l), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero; hence, an X exists such that

$$XX_1 = 0, \text{ only if } X_1 = 0;$$

C^j_r), right-hand division possible and unique in the subset E_j, that is, not every J is a right-hand divisor of zero; hence, a J exists such that

$$J_1 J = 0, \text{ only if } J_1 = 0;$$

C^j_l), left-hand division possible and unique in the subset E_j, that is, not every J is a left-hand divisor of zero; hence, a J exists such that

$$JJ_1 = 0, \text{ only if } J_1 = 0;$$

C^k_r), right-hand division possible and unique in the subset E_k, that is, not every K is a right-hand divisor of zero; hence, a K exists such that

$$K_1 K = 0, \text{ only if } K_1 = 0;$$

C^k_l), left-hand division possible and unique in the subset E_k, that is, not every K is a left-hand divisor of zero; hence, a K exists such that

$$KK_1 = 0, \text{ only if } K_1 = 0.$$

Conditions $A_1, A_2, A_3, A_4, A_5, A_6, A_7$, and A_8, together are equivalent to the associativity condition

$$A), \quad (X_1 X_2) X_3 = X_1 (X_2 X_3),$$

where X_p $(p = 1, 2, 3)$ are any numbers of the system E and where

$$X_p = J_p + K_p = \sum_{j_1=1}^{m} x_{pj_1} e_{j_1} + \sum_{k_1=m+1}^{n} x_{pk_1} e_{k_1}.$$

It was shown also that the conditions composing each definition are independent.

By systematic checking I have found that the ninety-six definitions given in the following tables, together with the seventy-eight of the former paper, give all possible methods of defining reducibility that depend upon the above twenty assumptions.

In this paper I use throughout the notation and the tables of the former paper; the definitions being based of course upon *Table I.* *Table II.* being complete, no additions are made; *Tables* III_2. and IV_2. of this paper supplement *Tables III.* and *IV.* of the former paper. In the notation R_{13_1}, the subscript indicates that this is the first definition which is to be inserted after R_{13} of the former paper to form the sequence as now completed. Definition R_{13_2} immediately follows R_{13_1}.

Independence Proofs.—With the exception of those indicated by a (*) in the tables, the conditions composing the various definitions are independent. The independence proofs are easily verified by *Table V.*, which is reprinted from the former paper.

§2.—*Definitions of Reducibility by Independent Assumptions.*

We reproduce *Table I.* from the former paper.

TABLE I.

Notation.	Assumptions.	Consequence.
D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r	C_1
D_2	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l	C_1
D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r	C_1
D_4	A_8, C^j_{kj}, C^k_{kj}, C^k_l	C_1
D_5	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r	C_2
D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l	C_3
D_7	A_3, C^j_{kj}, C^k_{kj} C^j_r	C_3
D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l	C_3
D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}
D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}
D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}
D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}

I.—From a consideration of this table, it is evident that D_6, D_5, D_2, D_1 are the only dependencies in which the division assumptions are on the system

E as a whole. There are eight possible combinations of these that give a definition of reducibility and they all appear in *Table II.*

II.—In D_{12}, D_{11}, D_{10}, D_9, D_8, D_7, D_4, and D_3 the division assumptions are on the subsets E_j and E_k. The thirty-eight definitions of *Table III.* with the additional twenty-four of *Table III$_2$.* give their sixty-two possible combinations.

TABLE III$_1$.

(1) Notation.	(2) From Table I.	(3) Assumptions.								Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{13_1}	D_{12}	A_5		C^j_{jk} C^k_{jk}			C^j_l			C^j_{kj}		
	D_{11}	A_6		C^j_{jk} C^k_{jk}				C^k_r		C^k_{kj}		
	D_7	A_8				C^j_r				" "		C_2
	....		C_1									
R_{13_2}	D_{12}	A_5		C^j_{jk} C^k_{jk}			C^j_l			C^j_{kj}		
	D_{11}	A_6		C^j_{jk} C^k_{jk}				C^k_r		C^k_{kj}		
	....		C_2									
	D_4	A_3							C^k_l	" "		C_1
R_{15_1}	D_{12}	A_5		C^j_{jk} C^k_{jk}			C^j_l			C^j_{kj}		
	....				C^k_{kj}							
	D_8	A_4		C^j_{jk} C^k_{jk}			C^j_l				C_2	
	D_4	A_3			C^k_{kj}				C^k_l	"		C_1
R_{17_1}	D_{12}	A_5		C^j_{jk} C^k_{jk}			C^j_l			C^j_{kj}		
	....				C^k_{kj}							
	D_7	A_8			C^k_{kj}	C^j_r				"		C_2
	D_4	A_3			C^k_{kj}				C^k_l	"		C_1
R_{17_2}	D_{12}	A_5		C^j_{jk} C^k_{jk}			C^j_l			C^j_{kj}		
	....				C^k_{kj}							
	D_7	A_8			C^k_{kj}	C^j_r				"		C_2
	D_8	A_7		C^j_{jk} C^k_{jk}				C^k_r			C_1	

TABLE III_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{17_3}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} C^j_r	"		C_2
	....	C_1			
R_{17_4}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_4	A_3 C^k_{kj} C^k_l	"		C_1
R_{19_1}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_4	A_3 C^j_{kj} C^k_l	"		C_1
R_{21_1}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	D_4	A_3 C^j_{kj} C^k_l	"		C_1
R_{21_2}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
R_{21_3}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	....	C_1			
R_{21_4}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_4	A_3 C^j_{kj} C^k_l	"		C_1

TABLE III$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{25_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_8	A_4 C^j_l	" "		C_2
	....	C_1			
R_{29_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	....	C_2			
	D_8	A_7 C^k_r	" "		C_1
R_{30_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_4	A_3 C^j_{kj} C^k_{kj} C^k_l		C_1	
R_{30_2}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_8	A_7 C^j_{jk} C^k_r	"		C_1
R_{30_3}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	....	C_1			
R_{31_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_8	A_7 C^j_{jk} C^k_r	"		C_1

TABLE III_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{33_1}	D_{10}	A_6 C^{j}_{kj} C^{k}_{kj} C^{ik}_{l}	C^{k}_{jk}		
	....	C^{ij}_{jk}			
	....	C_2			
	D_3	A_7 C^{ij}_{jk} C^{ik}_{r}	"		C_1
R_{34_1}	....	C^{k}_{jk}			
	D_9	A_5 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}	C^{ij}_{jk}		
	D_8	A_4 C^{k}_{jk} C^{ij}_{l}	"		C_2
	D_4	A_3 C^{j}_{kj} C^{k}_{kj} C^{ik}_{l}		C_1	
R_{34_2}	....	C^{k}_{jk}			
	D_9	A_5 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}	C^{ij}_{jk}		
	D_8	A_4 C^{k}_{jk} C^{ij}_{l}	"		C_2
	D_3	A_7 C^{ik}_{jk} C^{k}_{r}	"		C_1
R_{34_3}	....	C^{k}_{jk}			
	D_9	A_5 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}	C^{ij}_{jk}		
	D_8	A_4 C^{ik}_{jk} C^{ij}_{l}	"		C_2
	....	C_1			
R_{35_1}	....	C^{k}_{jk}			
	D_9	A_5 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}	C^{j}_{jk}		
	D_7	A_8 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}		C_2	
	D_3	A_7 C^{k}_{jk} C^{k}_{r}	"		C_1
R_{37_1}	....	C^{k}_{jk}			
	D_9	A_5 C^{j}_{kj} C^{k}_{kj} C^{ij}_{r}	C^{ij}_{jk}		
	....	C_2			
	D_3	A_7 C^{ik}_{jk} C^{k}_{r}	"		C_1

III.—The thirty-two definitions of *Table IV.* and the additional seventy-two of *Table IV_2.* each contain at least one division assumption upon the system E as a whole and at least one division assumption on one of its subsets.

TABLE IV_2.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
R_{54_1}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C_l	" "		C_2
	....	C_1			
R_{58_1}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C_r	" "		C_2
	....	C_1			
R_{58_2}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C_l	" "		C_1
R_{58_3}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_1	A_7 C^j_{jk} C^k_{jk} C_r	" "		C_1
R_{58_4}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_3	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1

TABLE IV$_3$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.					Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{58_5}	D_{12}	A_5	C^j_{jk} C^k_{jk}			C^j_l	C^j_{kj}		
	....			C^k_{kj}					
	D_8	A_4	C^j_{jk} C^k_{jk}			C^j_l		C_2	
	D_1	A_7	C^j_{jk} C^k_{jk}	C^k_{kj} C_r			“		C_1
R_{58_6}	D_{12}	A_5	C^j_{jk} C^k_{jk}			C^j_l	C^j_{kj}		
	....			C^k_{kj}					
	D_7	A_8		C^k_{kj}	C^j_r		“		C_2
	D_2	A_3	C^j_{jk} C^k_{jk}	C^k_{kj}	C_l		“		C_1
R_{58_7}	D_{12}	A_5	C^j_{jk} C^k_{jk}			C^j_l	C^j_{kj}		
	....			C^k_{kj}					
	D_7	A_8		C^k_{kj}	* C^j_r		“		C_2
	D_1	A_7	C^j_{jk} C^k_{jk}	C^k_{kj} C_r			“		C_1
R_{58_8}	D_{12}	A_5	C^j_{jk} C^k_{jk}			C^j_l	C^j_{kj}		
	....			C^k_{kj}					
	D_6	A_4	C^j_{jk} C^k_{jk}	C^k_{kj}	C_l		“		C_2
	D_4	A_8		C^k_{kj}		* C^k_l	“		C_1
R_{58_9}	D_{12}	A_5	C^j_{jk} C^k_{jk}			C^j_l	C^j_{kj}		
	....			C^k_{kj}					
	D_6	A_4	C^j_{jk} C^k_{jk}	C^k_{kj}	C_l		“		C_2
	D_8	A_7	C^j_{jk} C^k_{jk}			C^k_r		C_1	

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{12}	A_5 C_{jk}^j C_{jk}^k C_l^j	C_{kj}^j		
$R_{58_{10}}$	....	C_{kj}^k			
	D_6	A_4 C_{jk}^j C_{jk}^k C_{kj}^k C_l	"		C_2
	D_2	A_3 C_{jk}^j C_{jk}^k C_{kj}^k C_l	"		C_1
	D_{12}	A_5 C_{jk}^j C_{jk}^k C_l^j	C_{kj}^j		
$R_{58_{11}}$	....	C_{kj}^k			
	D_6	A_4 C_{jk}^j C_{jk}^k C_{kj}^k C_l	"		C_2
	D_1	A_7 C_{jk}^j C_{jk}^k C_{kj}^k C_r	"		C_1
	D_{12}	A_5 C_{jk}^j C_{jk}^k C_l^j	C_{kj}^j		
$R_{58_{12}}$	....	C_{kj}^k			
	D_6	A_4 C_{jk}^j C_{jk}^k C_{kj}^k C_l	"		C_2
	....	C_1			
	D_{12}	A_5 C_{jk}^j C_{jk}^k C_l^j	C_{kj}^j		
$R_{58_{13}}$	....	C_{kj}^k			
	D_5	A_8 C_{jk}^j C_{jk}^k C_{kj}^k C_r	"		C_2
	D_4	A_8 C_{kj}^k C_l^k	"		C_1
	D_{12}	A_5 C_{jk}^j C_{jk}^k C_l^j	C_{kj}^j		
$R_{58_{14}}$	....	C_{kj}^k			
	D_5	A_8 C_{jk}^j C_{jk}^k C_{kj}^k C_r	"		C_2
	D_3	A_7 C_{jk}^j C_{jk}^k $*C_r^k$		C_1	

TABLE IV_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
$R_{58_{15}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
$R_{58_{16}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1
$R_{58_{17}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	....	C_1			
$R_{58_{18}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
$R_{58_{19}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1

TABLE IV$_3$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{20}}$	D_8	A_4 C^j_{jk} C^k_{jk} $*C^j_l$		C_2	
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{21}}$	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_1	A_7 C^j_{ik} C^k_{jk} C^j_{kj} C_r	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{22}}$	D_7	A_3 C^j_{kj} C^j_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{23}}$	D_7	A_3 C^j_{kj} $*C^j_r$	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{24}}$	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_4	A_3 C^j_{kj} $*C^k_l$	"		C_1

TABLE IV$_3$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{25}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
$R_{58_{26}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_2	A_2 C^j_{jk} C^k_{ik} C^j_{kj} C_l	"		C_1
$R_{58_{27}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
$R_{58_{28}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	....	C_1			
$R_{58_{29}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{ik} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_4	A_3 C^j_{kj} C^k_l	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{20}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
$R_{58_{21}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_1	"		C_1
$R_{58_{22}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
$R_{58_{23}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	....	C_1			
$R_{58_{24}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_1	"		C_1

TABLE IV$_3$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3)		(6) Proved by (3) and (4).
			(4)	(5)	
	....	C_{kj}^{j}			
$R_{58_{36}}$	D_{11}	A_8 C_{jk}^{j} C_{jk}^{k} C_r^{k}	C_{kj}^{k}		
	....	C_2			
	D_1	A_7 C_{jk}^{j} C_{jk}^{k} C_{kj}^{j} C_r	"		C_1
	D_{10}	A_6 C_{kj}^{j} C_{kj}^{k} C_l^{k}	C_{jk}^{k}		
R_{66_1}	D_9	A_5 C_{kj}^{j} C_{kj}^{k} C_r^{j}	C_{jk}^{j}		
	D_6	A_4 C_{kj}^{j} C_{kj}^{k} C_l	" "		C_2
	....	C_1			
	D_{10}	A_6 C_{kj}^{j} C_{kj}^{k} C_l^{k}	C_{jk}^{k}		
R_{70_1}	D_9	A_5 C_{kj}^{j} C_{kj}^{k} C_r^{j}	C_{jk}^{j}		
	D_5	A_8 C_{kj}^{j} C_{kj}^{k} C_r	" "		C_2
	....	C_1			
	D_{10}	A_6 C_{kj}^{j} C_{kj}^{k} C_l^{k}	C_{jk}^{k}		
R_{70_2}	D_9	A_5 C_{kj}^{j} C_{kj}^{k} C_r^{j}	C_{jk}^{j}		
	....	C_2			
	D_2	A_3 C_{kj}^{j} C_{kj}^{k} C_l	" "		C_1
	D_{10}	A_6 C_{kj}^{j} C_{kj}^{k} C_l^{k}	C_{jk}^{k}		
R_{70_3}	D_9	A_5 C_{kj}^{j} C_{kj}^{k} C_r^{j}	C_{jk}^{j}		
	....	C_2			
	D_1	A_7 C_{kj}^{j} C_{kj}^{k} C_r	" "		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{70_4}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} $*C^j_l$	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
R_{70_5}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
R_{70_6}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_3	
	D_3	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
R_{70_7}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}		C_3	
	D_7	A_8 C^j_{kj} C^k_{kj} $*C^j_r$			
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
R_{70_8}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_4	A_3 C^j_{kj} C^k_{kj} C^k_l		C_1	

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
R_{70_9}	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_r	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{10}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{11}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{12}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	....	C_1			
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{13}}$	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_4	A_8 C^j_{kj} C^k_{kj} C^k_l		C_1	

TABLE IV₂—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
$R_{70_{14}}$	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_3	A_7 C^j_{jk} $*C^k_r$	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
$R_{70_{15}}$	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
$R_{70_{16}}$	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{17}}$	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	....	C_1			
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{18}}$	....	C_2			
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
$R_{70_{19}}$	....	C^j_{jk}			
	....	C_2			
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
$R_{70_{20}}$	D_8	A_4 C^k_{jk} $*C^j_l$	"		C_2
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
$R_{70_{21}}$	D_8	A_4 C^k_{jk} C^j_l	"		C_2
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
$R_{70_{22}}$	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
$R_{70_{23}}$	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1

TABLE IV$_3$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{24}}$	D_6	A_4 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_2
	D_4	A_8 C_{kj}^j C_{kj}^k *C_l^k		C_1	
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{25}}$	D_6	A_4 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_2
	D_8	A_7 C_{jk}^k C_r^k	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{26}}$	D_6	A_4 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_2
	D_2	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{27}}$	D_6	A_4 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_2
	D_1	A_7 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{28}}$	D_6	A_4 C_{jk}^k C_{kj}^j C_{kj}^k C_l	"		C_2
	....	C_1			
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{29}}$	D_5	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_2
	D_4	A_2 C_{kj}^j C_{kj}^k C_l^k		C_1	

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{20}}$	D_5	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_2
	D_3	A_7 C_{jk}^k $*C_r^k$	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{21}}$	D_5	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_2
	D_2	A_3 C_{jk}^k C_{kj}^j C_{kj}^k C_i	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{22}}$	D_5	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_2
	D_1	A_7 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{23}}$	D_5	A_8 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_2
	....	C_1			
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{24}}$	....	C_2			
	D_2	A_3 C_{jk}^k C_{kj}^j C_{kj}^k C_i	"		C_1
	....	C_{jk}^k			
	D_9	A_5 C_{kj}^j C_{kj}^k C_r^j	C_{jk}^j		
$R_{70_{25}}$	....	C_2			
	D_1	A_7 C_{jk}^k C_{kj}^j C_{kj}^k C_r	"		C_1

§3.—*Independence Proofs.*

The sets of conditions in *Tables* III_2. and IV_2. yield reducibility. It can readily be seen that there are not enough assumptions in any one of these sets to prove the associativity A of the system E. In each case this can be accomplished by adding the assumptions A_1 and A_2. By the use of *Table* V. the mutual independence of the conditions composing each of the definitions of *Table* III_2. and those of all but twelve of the definitions of *Table* IV_2. can be established. The independencies of *Table* V. are based upon the following multiplication tables, for which the borders have been omitted.

I.

e_2	0	0
0	e_1	0
0	0	e_3

$E_j = e_1, e_2$; $E_k = e_3$.

II.

$e_1 + e_2$	0
0	e_2

$E_j = e_1$; $E_k = e_2$.

III.

e_1	0	e_1
0	e_2	0
0	0	e_3

$E = e_1, e_2$; $E_k = e_3$.

IV.

e_1	0	0
e_1	e_2	0
0	0	e_3

$E_j = e_1$; $E_k = e_2, e_3$.

V.

e_1	e_1
e_1	e_2

$E = e_1$; $E_k = e_2$.

VI.

0	0	e_1
e_1	e_2	0
0	0	e_3

$E_j = e_1, e_2$; $E_k = c_3$.

VII.

e_1	0	e_3
0	e_2	0
0	e_3	0

$E_j = e_1$; $E_k = e_2, e_3$.

VIII.

e_1	0	0
e_2	0	0
0	0	e_3

$E_j = e_1, e_2$; $E_k = e_3$.

IX.

0	0	0
e_1	e_2	0
0	0	e_3

$E_j = e_1, e_2$; $E_k = e_3$.

TABLE V.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	C_1	C_2	C^j_{jk}	C^k_{jk}	C^j_{kj}	C^k_{kj}	C_l	C_r	C^j_l	C^k_l	C^j_r	C^k_r	Proof.
1	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	I.
2	★	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (1).
3	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	★	II.
4	★	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (3).
5	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	★	III.
6	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	★	IV.
7	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	Interchange j, k in (6).
8	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	Interchange j, k in (5).
9	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	★	V.
10	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	VI.
11	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	Interchange j, k in (9).
12	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	VII.
13	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	★	★	★	Interchange j, k in (12).
14	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	Interchange j, k in (10).
15	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	VIII.
16	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	★	Interchange j, k in (15).
17	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	IX.
18	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	Interchange j, k in (17).

§4.—*Dependence Proofs.*

The twelve sets of conditions of *Table* IV_2. that require individual consideration are R_{58_7}, R_{58_8}, $R_{58_{14}}$, $R_{58_{20}}$, $R_{58_{22}}$, $R_{58_{24}}$, R_{70_4}, R_{70_7}, $R_{70_{14}}$, $R_{70_{20}}$, $R_{70_{24}}$, and $R_{70_{30}}$.

1. That C^j_r is a consequence of the other conditions of R_{58_7} can be shown in the following manner. The conditions from which C^j_{kj} is derived in D_{12} are independent. By D_1 and D_5 it is seen that C_1 and C_2 are consequences of A_7, A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r and all these conditions are mutually independent. According to the condition C_r, there exist a J and a K such that $(J_1+K_1)(J+K)=0$,

only if $J_1 = 0 = K_1$. Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj} that

$$J_1 J + K_1 K = 0, \text{ only if } J_1 = 0 = K_1$$

or

$$J_3 + K_3 = 0, \qquad \text{by } C_1, C_2.$$

By the linear independence of the units, it follows that

$$\left.\begin{array}{l} J_3 = J_1 J = 0 \\ K_3 = K_1 K = 0 \end{array}\right\} \text{only if } \left\{\begin{array}{l} J_1 = 0 \\ K_1 = 0. \end{array}\right.$$

The former condition is C^j_r.*

2. That C^k_l is a consequence of the other conditions of R_{58_8} can be shown by the successive use of D_{12}, D_6, D_2, and C_l.

3. That C^k_r is a consequence of the other conditions of $R_{58_{14}}$ can be shown by the successive use of D_{13}, D_5, D_1, and C_r.

4. That C^j_l is a consequence of the other conditions of $R_{58_{20}}$ can be shown by the successive use of D_{11}, D_6, D_2, and C_l.

5. That C^j_r is a consequence of the other conditions of $R_{58_{22}}$ can be shown by the successive use of D_{11}, D_5, D_1, and C_r.

6. That C^k_l is a consequence of the other conditions of $R_{58_{24}}$ can be shown by the successive use of D_{11}, D_6, D_2, and C_l.

7. That C^j_l is a consequence of the other conditions of R_{70_4} can be shown by the successive use of D_{10}, D_6, D_2, and C_l.

8. That C^j_r is a consequence of the other conditions of R_{70_7} can be shown by the successive use of D_{10}, D_5, D_1, and C_r.

9. That C^k_r is a consequence of the other conditions of $R_{70_{14}}$ can be shown by the successive use of D_{10}, D_5, D_1, and C_r.

---

*In §5 of the previous paper the corresponding dependence theorems are correctly given. Some of the proofs, however, contain a weakness. Thus, in the above proof, it is not correct to continue the argument

Multiplying on the left by J', since

$$J_1 J + K_1 K = 0, \text{ only if } J_1 = 0 = K_1, \text{ and since } J_1 J = J_3 \text{ and } K_1 K = K_3, \text{ then}$$
$$J'(J_1 J + K_1 K) = 0, \text{ only if } J_1 = 0 = K_1 \text{ and therefore}$$
$$J'(J_1 J + K_1 K) = 0, \text{ only if } J_1 J = 0 = K_1 K,$$

which gives the required condition.

The reason that this argument is not valid is that the *product* $J_3\ (= J_1 J)$, although different from zero, may be a right-hand divisor of zero.

By the method used in the body of this paper all the dependence proofs in the previous paper can easily be modified so as to be correct.

10. That C_l^l is a consequence of the other conditions of $R_{70_{20}}$ can be shown by the successive use of D_9, D_6, D_2, and C_l.

11. That C_l^k is a consequence of the other conditions of $R_{70_{24}}$ can be shown by the successive use of D_9, D_6, D_2, and C_l.

12. That C_r^k is a consequence of the other conditions of $R_{70_{30}}$ can be shown by the successive use of D_9, D_5, D_1, and C_r.

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CONTENTS.

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